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The Smoluchowski Limit for a Simple Mechanical Model

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We consider a vertical stick constantly accelerated along the x-axis by a force F and which elastically collides with point particles of the same mass (atoms). The atoms are initially Poisson distributed and are allowed to have four velocities only. It is shown that under suitable scaling of the system the displacement $Q(t)$ of the stick satisfies a nontrivial CLT: $Q(t) = vFt + D^{1/2}W(t)$ (Smoluchowski equation), where the values of v and D depend on the fact that one atom may collide several times.

KEY WORDS: Invariance principle; Newtonian evolution; Poisson point processes.

1. INTRODUCTION

Consider a Brownian particle subject to an external force F. In the so-called high-friction limit its displacement $Q(t)$ is described by the Smoluchowski equation[1,2]

$$dQ^*(t) = vF(Q^*(t))\,dt + D^{1/2}\,dW(t) \tag{1.1}$$

$v, D > 0$, $W(t)$ a standard Wiener process.

In thermal equilibrium it is generally believed that the Einstein relation $v = D/2KT$ (K is Boltzmann's constant, T is the absolute temperature), holds.[8]

Our aim is to derive (1.1) from first principles, i.e., if $Q_A(t) = Q(At)/\sqrt{A}$, $t \geqslant 0$, $Q(t)$ denoting the position of a test particle subject to an

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external force $F_A = F/\sqrt{A}$ interacting with a heat bath, then we wish to show that $Q_A \equiv \{Q_A(t)\}_{t \geqslant 0}$ converges in law to $Q^* \equiv \{Q^*(t)\}_{t \geqslant 0}$, as $A \to \infty$, given by (1.1) with v and D satisfying the Einstein relation.

The simplest model one might think of is the one-dimensional infinite ideal gas of identical point particles (atoms) of mass $m = 1$ with velocities $\pm v$ each with probability 1/2. The test particle (molecule) is an extra particle identical to the others, placed initially at the origin, say, with velocity $V = \pm v$. The system then evolves according to Newtonian dynamics with elastic collisions between the atom and the molecule. The molecule is also subject to a constant for $F_A = (1/\sqrt{A} F$. Then

$$Q_A(t) \approx \frac{1}{\sqrt{A}} \sum_{i=1}^{[At]} V_i \Delta t_i + \frac{1}{\sqrt{A}} \sum_{i=1}^{[At]} \frac{1}{2} F \Delta t_i^2 \tag{1.2}$$

where $V_i t_i + (F/2\sqrt{A})\, \Delta t_i^2$ gives the increments between collisions, V_i being the velocity of the molecule between the ith and the next collision and Δt_i the time in between. If the increments were independent, the first term on the rhs of (1.2) would converge to a Wiener process and the second would converge to the drift $\sim Ft$. We expect therefore that in this scaling Q_A converges in law to a Smoluchowski process with the "Einstein relation" $\frac{1}{2} D/v = u^2$, defining the "temperature" of the bath by the mean kinetic energy of an atom.

Surprisingly, we found that this is not the case,[4] but that Q_A simply converges to a Wiener process with the diffusion constant v/ρ, ρ being the initial density of the bath. This is the process one obtains for $F = 0$. A little analysis of the motion shows that first of all the increments in (1.2) are not independent, due to recollisions (see Fig. 1, Section 3). Then our result shows that their effect persists also in the limit $A \to \infty$. What happens is roughly this: If the molecule has, due to the force, a drag to the right, then atoms pile up in front of it and counteract the drag. We treat here a two-dimensional system of colliding equal masses because the result is mathematically more interesting: In the limit we obtain at least a dissufion with a drift. The molecule is now a stick of length l of fixed orientation orthogonal to the x axis which can only move along the x axis. The bath is initially a Poisson gas of atoms which are allowed to have four velocities $(\pm v_1, \pm v_2)$ only. Note that in collisions only the x components of the velocities are exchanged. Since $v_2 \neq 0$, every atom can ony interact for a time l/v_2 with the stick; the possibility of recollisions is reduced, but, as in one dimension, the effects or recollision will persist in the limit $A \to \infty$.

[4] Unpublished results. One may base a simple proof on a "noncrossing" argument used in ref. 9.

(In fact, slightly modifying our proof to fit the easier one-dimensional model described before, one obtains the quoted result.) Also, an "Einstein relation" does not hold.

It is probably fair to say that our results and analysis of the motion need to be understood for treating models for which the Einstein relation will hold, i.e., for systems in equilibrium prescribing, e.g., Maxwellian velocities to the atoms. However, it will not be possible to generalize our analysis, which looks very closely at the details of collision sequences, to models in which the atoms are allowed to have arbitrarily small velocities as in the Maxwellian case.

It should be noted that here the heuristic argument following (1.2), assuming independence properties, only intends to give a rough idea of why a diffusion process should appear in the above scaling limit.

To conclude, we note that the scaling described above is equivalent to the following one, which we find more convenient and which is used in the paper. Scaling space as $1/\sqrt{A}$ has the effect of scaling $\rho \sim A$, $l \sim 1/\sqrt{A}$ in two dimensions and since the mean collision time ($F=0$) is given by $\theta = 1/\rho l v_1$, looking at times At is equivalent to setting $\theta \sim A^{-1}$, and thus $v_i \sim \sqrt{A}$, $i = 1, 2$.

After this work was completed, the Smoluchowski limit and Einstein relation were proven for the much easier model of an infinite harmonic chain.[10]

2. THE MODEL AND THE RESULT

We describe first the "heat bath." Consider on $\mathbb{R}^4$ a (position, velocity) point process $(\Omega, \mathscr{F}, P)$ with intensity measure μ_A on the Borel algebra $\mathscr{B}(R^4)$ given by (for simplicity of notation we suppress the index A)

$$d\mu_A(\mathbf{q}, \mathbf{p}) = \rho d^2\mathbf{q} \cdot \frac{1}{4}\left[\sum_{i=0}^{3} \delta_{\mathbf{v}_i}(d^2\mathbf{p})\right], \qquad \rho = \rho_0 A \tag{2.1}$$

$d^2\mathbf{q}$ is the Lebesgue measure on $\mathbb{R}^2$ and $\delta_{\mathbf{v}_i}(d^2\mathbf{p})$, $i = 0,..., 3$, is the Dirac measure concentrated on

$$\mathbf{v}_0 \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \mathbf{v}_1 \equiv \begin{pmatrix} -v_1 \\ v_2 \end{pmatrix}, \qquad \mathbf{v}_2 \equiv \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}, \qquad \mathbf{v}_3 \equiv \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$$
$$v_1 = A^{1/2} v^1, \qquad v_2 = A^{1/2} v^2, \qquad v^1, v^2 > 0 \tag{2.2}$$

A configuration $(\mathbf{q}_j, \mathbf{p}_j)_{j \in N}$ (positions and velocities) of identical point particles (atoms) of mass $m = 1$ of the heat bath is a realization of $(\Omega, \mathscr{F}, P)$ so that the positions are uniformly distributed with density ρ and the velocity of each particle is independently chosen to be $\mathbf{v}_i$, $i = 0,..., 3$, with probability 1/4, i.e.,

$$\mathbf{p}_j = \begin{pmatrix} p_j^1 \\ p_j^2 \end{pmatrix} \in \{\mathbf{v}_i\}, \qquad i = 0,..., 3$$

This describes the initial state of the heat bath at time $t = 0$.

We now put a vertical stick of mass $m = 1$ and length $l = l_0/\sqrt{A}$ with its center at the origin of the system with velocity

$$\mathbf{V}_A = \pm \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$$

The stick cannot rotate and can only move along the x axis. The stick is also subject to a force $\mathbf{F} = \binom{f}{0}$, $f = f_0 A$. One might of course also think of other scalings where $v \sim \sqrt{A}$ and $\rho l \sim A$; ρ and l are scaled appropriately. From our analysis, however, it may be easily seen that if $l \sim A^{-\gamma}$, $\gamma < 1/2$, the limit equation has no drift and a drift appears only if $\gamma \geqslant 1/2$.

We now let the system evolve in time t.

The atoms interact with the stick through elastic collisions, but do not interact with each other. The elastic collision between, say, atom (i) and the stick happens if at some time $\tau q_i^1(\tau) = Q_A(\tau)$, $|q_i^2(\tau)| < l$ [$Q_A(\tau)$ denotes the position of the center of the stick on the x axis at time τ] and it is defined as follows.

If $V(\tau^-)$ and $\mathbf{p}_i(\tau^-)$ are the precollision velocities of the stick and of the atom (i), then

$$\mathbf{V}_A(\tau^+) = \begin{pmatrix} p_i(\tau^-) \\ 0 \end{pmatrix}, \qquad \mathbf{p}_i(\tau^+) = \begin{pmatrix} V_A^1(\tau^-) \\ p_i^2 \end{pmatrix} \tag{2.3}$$

are the postcollision velocities of the stick and the atom. Note that p_i^2 does not change with them.

Between collisions each particle moves freely with constant velocity $\mathbf{p}$ and the position of the stick is given by

$$Q_A(t) = Q_A(\tau) + V_A^1(\tau)(t - \tau) + \tfrac{1}{2} f_0 A (t - \tau)^2 \tag{2.4}$$

where τ is the last collision time before t or $\tau = 0$.

The evolution is, of course, not well defined for all initial configurations of the heath bath.

Problematic events are infinitely many collisions of the stick in a finite amount of time and multiple collisions (i.e., two or more particles collide with the stick at the same time). One needs to show that those bad events have measure zero with respect to P, so that there exists $\Omega' \subset \Omega$ with $P(\Omega') = 1$ and on Ω' the evolution of the system is well defined by the above prescription for any $t < +\infty$. We were not able to find a quick proof of this, but Shigeo Kusuoka has a very original way to deal with this problem. Space limitations preclude our giving his proof here; it is available from us upon request. Therefore, we simply assume that the dynamics is defined for all $t \geq 0$.

We denote by $\{V_A(t)\}_{t \in [0, T]}$ the right continuous velocity process of the stick. We shall prove the following.

Theorem 2.1. For any $T < +\infty$, the process $Q_A \equiv \{Q_A(t)\}_{t \in [0, T]}$ converges, as $A \to \infty$, in distribution to a diffusion process $Q^* \equiv \{Q^*(t)\}_{t \in [0, T]}$ given by the stochastic differential equation

$$dQ^*(t) = \frac{1}{2} f_0 \theta \frac{1 - e^{-r}}{r} dt + (v^1 q)^{1/2} \, dW(t) \tag{2.5}$$

where $W(t)$ is a standard Wiener process and $\theta = (\rho_0 l_0 v^1)^{-1}$ is the average time between collisions, $r = \rho_0 l_0^2 v^1 / v^2$ is the interaction time measured in units of θ, and $q = (\rho_0 l_0)^{-1}$ is the mean free path.

Remark 2.1. Since T is arbitrary, we obtain the result for any $t \in [0, \infty)$.[4]

Remark 2.2. Keep q fixed and let $l_0 \to \infty$ and observe that the drift of $Q^*(t)$ goes to zero. This is what we obtain in the one-dimensional model described in the Introduction.

Keep q fixed and let $l_0 \to 0$; then we obtain

$$dQ^*(t) = \tfrac{1}{2} f_0 \theta \, dt + (v^1 q)^{1/2} \, dW(t)$$

This is what one obtains using a Poisson bath all times.

The proof of the theorem is done in two "standard" steps: (i) Prove that some easy to handle process (which, however, has the main features of the true process) converges in distribution to the limit (2.5); (ii) Prove that the true process and the process of (i) are close in probability. In this step we use, of course, a coupling argument. The theorem follows from (i) and (ii).[5]

3. THE MECHANICAL MOTION AND TECHNICALITIES

Given a trajectory of the stick, let $\{S_k\}_{k=0,\dots,N}$ be the sequence of collision times and of the times in which the velocity is equal to v_1 and let $V_k = V_A(S_k^+)$, $k = 0,\dots, N$, be the postcollision velocity of the stick. From

$\{S_k, V_k\}_{k=0,\ldots,N}$ we now construct a sequence $\{\Delta_{A,j}, Q_{A,j}\}_{j=0,\ldots,N_A}$ of intervals and increments.

Figure 1 (3.6) shows a typical collision sequence considered throughout the paper. The reader may consult the figure for the following definitions. Collision events considered later are also shown in the figure and the reader should refer to it for clarification. For

$$S_{k_0} \begin{cases} 0 & \text{if } \; V_A(0) < 0 \\ \inf\{S_k \geqslant 0 \colon V_k = -v_1\} \end{cases} \tag{3.1}$$

set

$$\Delta^+_{A,0} = (0, S_{k_0}], \qquad \tau^+_{A,0} = S_{k_0} \tag{3.2}$$

For all $j \geqslant 1$ let

$$S_{k_{2j-1}} = \sup\{S_k \geqslant S_{k_{2(j-1)}} \colon V_h < 0 \quad \text{for all} \quad k_{2(j-1)} \leqslant h \leqslant k\} \tag{3.3}$$

and

$$S_{k_{2j}} = \inf\{S_k > S_{k_{2j-1}} \colon V_k = -v_1 \quad \text{and} \quad V_{k-1} > 0\} \tag{3.4}$$

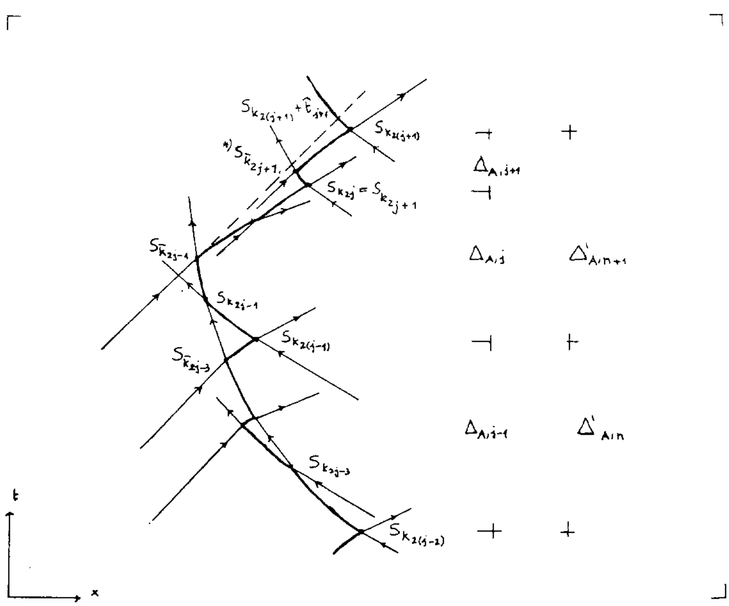

Fig. 1. The bold line indicates the displacement of the stick. The thin line traces the position of the colliding atoms. (*) Collisions with "shadow" atoms are not possible in the one-dimensional model.

Observe that $S_{k_{2j}}$ is a Markov time. The increments we wish to consider are

$$\Delta_{A,j} = (S_{k_{2(j-1)}}, S_{k_{2j}}] \tag{3.5}$$

with lengths

$$\tau_{A,j} = S_{k_{2j}} - S_{k_{2(j-1)}} \tag{3.6}$$

For the analysis to follow we found it useful to split $\Delta_{A,j}$ into two intervals, $\Delta_{A,j}^{-}$ and $\Delta_{A,j}^{+}$. The $\Delta_{A,j}^{-}$ should be thought of as the time interval in which the stick moves mainly with negative velocity: in $\Delta_{A,j}^{-}$ one might find "short" periods of time during which the stick moves with positive velocity and which end in recollisions with atoms having negative velocities. This is specified in the following definition, where we introduce the non-Markov time $S_{\tilde{k}_{2j-1}} \in (S_{k_{2j-1}}, S_{k_{2j}})$:

$$S_{\tilde{k}_{2j-1}} = \inf\{S_k > S_{k_{2j-1}} \colon V_k = v_1 \qquad \text{and} \qquad \inf\{S_{k'} > S_k \colon V_{k'} < 0\} = S_{k_{2j}}\} \tag{3.7}$$

hence

$$\begin{aligned} \Delta_{A,j}^{-} &= (S_{k_{2(j-1)}}, S_{\tilde{k}_{2j-1}}]; \qquad & \Delta_{A,j}^{+} &= (S_{\tilde{k}_{2j-1}}, S_{k_{2j}}] \\ \tau_{A,j}^{-} &= S_{\tilde{k}_{2j-1}} - S_{k_{2(j-1)}}, \qquad & \tau_{A,j}^{+} &= S_{k_{2j}} - S_{\tilde{k}_{2j-1}} \end{aligned} \tag{3.8}$$

From this we now construct the position increments.

For all $j \geqslant 0$, let

$$\begin{gathered} Q_{A,j}^{-} = Q_A(S_{\tilde{k}_{2j-1}}) - Q_A(S_{k_{2(j-1)}}), \qquad Q_{A,j}^{+} = Q_A(S_{k_{2j}}) - Q_A(S_{\tilde{k}_{2j-1}}) \\ Q_{A,j} = Q_{A,j}^{-} + Q_{A,j}^{+} \end{gathered} \tag{3.9}$$

For all $t \in [0, T]$ set

$$N_A(t) = \begin{cases} \sup\{k \geqslant 0 \colon \sum_{j=0}^{k} \tau_{A,j} \leqslant t\} & \text{if} \quad \tau_{A,0}^{+} \leqslant t \\ 0 \quad \text{otherwise} \end{cases} \tag{3.10}$$

Clearly, the increments $Q_{A,j}$ are not independent, nor do they constitute a Markov process, because of the possibility of recollisions. Our aim is to show that the increments are weakly dependent; for this we clearly need a control on recollisions. Note that the stick (through the force field and through recollisions) could become very slow, and slow velocities increase the possibility or recollisions. Our estimate will be based on the following definition.

Definition 3.1. For $j<k$, we shall say that the increments $Q^-_{A,j}$ and $Q^-_{A,k}$ are connected [denoted by $j(\mathrm{c})\,k$] iff there exists an atom that collides during $\Delta^-_{A,j}$ and $\Delta^-_{A,k}$, and $\Delta^-_{A,k}$ is the first interval after $\Delta^-_{A,j}$ in which the atom collides. Moreover, we shall say that $Q^-_{A,j_1},..., Q^-_{A,j_n}$, $j_1<j_2<\cdots<j_n$, are connected if there exists a subset $\{j_{n_1},..., j_{n_k}\}$ of $\{j_1,..., j_n\}$ such that

$$j_{n_1}=j_1, \qquad j_{n_k}=j_n, \qquad j_{n_i}<j_{n_{i+1}}, \qquad j_{n_i}(\mathrm{c})\,j_{n_{i+1}} \qquad \text{for all} \quad i=0,..., k \tag{3.11}$$

Lemma 3.1 (Key estimate). For all $T<+\infty$, let $\Omega^-_A(T)$ be the set of the trajectories in $[0, T]$ of the mechanical stick for which no more than three increments are connected. Then there exists $\beta>0$ such that

$$P(\Omega^-_A(T)^c)<CA^{-\beta}$$

Proof. In all the proofs we denote by C_i and D_j finite and positive constants. The indices i and j distinguish possibly different constants. The constants D_j for the same j may occur in different proofs, but then the constants may be different.

Let j_0 be the index of the first increment which is connected with at least three increments in the future. Then

$$\Omega^-_A(T)^c=\bigcup_{j=1}^{N_A(T)-3}\{j=j_0\} \tag{3.12}$$

Thus, to prove Lemma 3.1, it is enough to show that for some v, $C_0>0$,

$$P(\{N_A(t)>vAt\})\leqslant C_0A^{-1} \tag{3.13}$$

and that

$$P(\{j=j_0\})\leqslant C_1A^{-(1+\beta)} \tag{3.14}$$

The following simple lemma provides (3.13).

We shall call an atom "fresh" if it did not collide in the past.

Lemma 3.2. Let $\{S'_n\}_{n=0,...,N}$ be the subsequence of $\{S_k\}_{k=0,...,N}$ of the collision times with fresh atoms. Then for every $\alpha>0$ and A sufficiently large

$$P(\{\exists n\colon S'_n-S'_{n-1}>A^{-1+\alpha}\})\leqslant CAT\exp(-\theta^{-1}A^\alpha) \tag{3.15}$$

for a finite constant C. Moreover, (3.13) holds for $v>3\theta^{-1}$ and $C_0=6\theta^{-1}/(v-3\theta^{-1})^2\,T^2A$.

Proof. Let $[0, T] = \bigcup_{i=0}^{M} \Gamma_i$, $\{\Gamma_i\}_{i=0,\dots,M}$, be a sequence of non-overlapping intervals of length $A^{-1}/2 \leqslant |\Gamma_i| \leqslant A^{-1}$. Then, for A large enough,

$$\begin{aligned}
&\{\exists n\colon S'_n - S'_{n-1} > A^{-1+\alpha}\} \\
&\leqslant \bigcup_{i=0}^{M-[A^\alpha]} \left\{ \text{during } \bigcup_{j=i}^{i+[A^\alpha]} \Gamma_j \text{ the stick has no collisions} \right. \\
&\qquad\qquad \left. \text{with fresh atoms} \right\} \\
&= \bigcup_{i=0}^{M-[A^\alpha]} N_i
\end{aligned}$$

Observing the strong Markov property of the Poisson field, $R_A^{\pm} = \frac{1}{2}\rho l\,|\pm v_1 - V_A(t)|$ appears as the equilibrium rate for collisions with fresh atoms given that the stick has velocity $V_A(t)$. We note first that

$$\bar{R}_A(S, T) = \sup_{u \in [S, S+T]} R_A^{\pm}(u) \leqslant \rho l(2v_1 + fT) \tag{3.16}$$

and

$$\underline{R}_A(S, T) = \min\Big(\inf_{\substack{u \in [S, S+T] \\ V_A(u^-) < 0}} R_A^{+}(u), \inf_{\substack{u \in [S, S+T] \\ V_A(u^-) > 0}} R_A^{-}(u)\Big) \geqslant \rho l v_1 = \theta^{-1} A \tag{3.17}$$

and hence estimates involving collisions with fresh atoms may be bounded by estimates on events related to Poisson processes with intensity measure $\lambda \geqslant \bar{R}_A(S, T)$ or $\lambda \leqslant \underline{R}_A(S, T)$. We shall denote the corresponding probabilities by λ-P, so that it is understood that in λ-$P(N)$ the event N refers to the Poisson process with intensity λ. Such estimates are referred to as "Poisson domination."

Note, however, that the stick accelerates when moving to the right to speeds greater than v_1, thus creating a shadow, i.e., a nonequilibrium distribution of fresh atoms to the left of it. In the one-dimensional setting this shadow is a pure vacuum. In the case at hand all atoms relevant to the motion of the stick have to be in the strip of width l (these are real and virtual atoms, i.e., those which, due to the motion, of the stick are not there). Since $v_2 > 0$ they are renewed every time l/v_2.

Let

$$\bigcup_{j=i}^{i+|A^\alpha|} \Gamma_j = [t_i, t_{i+1})$$

then on N_i there are no more collisions during $[t_i + l/v_2, t_{i+1})$. By time $t_i + l/v_2$ the stick moves either to the left or to the right. In the latter case it sees fresh atoms with $p^1 = -v_1$ at the equilibrium rate. In the former case it sees fresh atoms with $p^1 = v_1$ at the equilibrium rate by time $t_i + 2l/v_2$, since it takes time l/v_2 to replace a shadow.

Then for all $i = 0,..., M - [A^\alpha]$

$$P\left(\left\{\text{during the interval } \bigcup_{j=i}^{i+[A^\alpha]} \Gamma_j \text{ the stick has no collisions with fresh atoms of opposite velocity}\right\}\right)$$

$$\leqslant \theta^{-1}A - P(\{\text{no points within } [t_i + 2l/v_2 \cdot t_{i+1}]\})$$

$$< C_1 \exp(-\theta^{-1}A^\alpha) \tag{3.18}$$

with an appropriate constant C_1. ∎

To prove (3.13), observe that

$$\{\sup_{s\in[0,T]} \bar{R}_A(s, T) > 3\theta^{-1}A\} \subseteq \{\exists n: S'_n - S'_{n-1} > v_1/f\}$$

and since $v_1/f = (v^1/f_0)\, A^{-1/2}$, by virtue of (3.15), we need only estimate

$$P(\{N_A(T) > vAT\} \cap \{\sup_{s\in[0,T]} \bar{R}_A(s, T) \leqslant 3\theta^{-1}A\})$$

Since $N_A(T)$ indicates the number of increments $\tau_{A,j}$ (starting at collisions with fresh atoms), then

$$N_A(T) \leqslant \#\ \text{of collisions with fresh atoms in } [0, T]$$

and

$$P(\{N_A(T) > vAT\} \cap \{\sup_{s\in[0,T]} \bar{R}_A(s, T) \leqslant 3\theta^{-1}A\}$$

$$\leqslant 3\theta^{-1}A - P(\{\text{the number of points in } [0, T] \geqslant vAT\})$$

$$\leqslant 3\theta^{-1}/(v - 3\theta^{-1})^2\, T^2 A$$

Remark 3.1. Given $0 < \alpha < 1$, set $G_A^\alpha = \{\text{for all } n,\ S'_n - S'_{n-1} \leqslant A^{-1+\alpha}\}$; then from (3.15) it follows that $(G_A^\alpha)^c$ has exponentially small probability. ∎

To prove (3.14) note that

$$\{j = j_0\} = (\{j = j_0\} \cap \{j(\mathrm{c})(j+k) \text{ for some } k \geqslant 3\}) \cup D(j) \tag{3.19}$$

where

$$D(j) = \{j = j_0\} \cap \left[\bigcup_{n=1,2} \{j(\mathrm{c})(j+n) \text{ and } (j+n)(\mathrm{c})(j+k) \text{ for some } k \geqslant 3\} \right.$$
$$\left. \cup \{j(\mathrm{c})(j+1),\ (j+1)(\mathrm{c})(j+2), \text{ and } (j+2)(\mathrm{c})(j+k) \text{ for some } k \geqslant 3\} \right] \tag{3.20}$$

Then (3.14) will follow from

step 1: $P(\{j=j_0\} \cap \{j(\mathrm{c})\, j+k \text{ for some } k \geqslant 3\}) < C_2 A^{-(1+\beta)}$
step 2: $P(D(j)) \leqslant C_3 A^{-(1+\beta)}$

The key observation made in both steps is the following: by definition of j_0, no increment $Q^-_{A,n}$ with $n < j_0 - 1$ can be connected with $Q^-_{A,k}$ for $k \geqslant j_0$. The increment Q^-_{A,j_0-1} can only be connected with Q_{A,j_0+1} iff Q^-_{A,j_0+1} is not connected with $Q^-_{A,k}$ for all $k > j_0 + 1$.

The proof of step 1 contains all the essential ingredients, which is simple combinations provide also the proof of step 2.

Proof of Step 1. By the observation above on $\{j = j_0\} \cap \{j(\mathrm{c})(j+k)$ for some $k \geqslant 3\}$ there exists a fresh atom collides first during $\Delta^-_{A,j}$ and recollides for the first time during $\Delta^-_{A,k}$. We distinguish two cases:

(i) The precollision velocity of the atom before the first collision with the stick is $p^1_{(j)} = v_1$. Then the atom in all subsequent collisions is to the left of the stick. We have the following trivial bound on the postcollision velocity $p^{-1}_{(j)}$ of the atom after its first collision:

$$p^{-1}_{(j)} \leqslant -v_1 + f\tau^-_{A,j} \tag{3.21}$$

where the equal sign holds if there is no other collision during $\Delta^-_{A,j}$, i.e., the first collision of the atom is at $S_{k_{2j-1}}$.

Note that in all future recollisions the 1-component of the velocity of the atom decreases, so that the right-hand side of (3.21) is still a bound for all the pre- and postcollision velocities of the atom.

Let s be the time of the first collision of the atom. After $s + l/v_2$ the atom cannot be reached anymore by the stick. If a recollision is to happen at time $s + u$, then by virtue of (3.21)

$$\int_s^{s+u} V(t)\, dt \leqslant (-v_1 + f\tau^-_{A,j})\, u$$

or

$$\int_s^{s+u} [V(t) + v_1]\, dt \leqslant f\tau^-_{A,j} u \tag{3.22}$$

with

$$u \leqslant l/v_2 \tag{3.23}$$

Now, since the atom can only change its velocity due to collisions with the stick and since each increment $Q^-_{A,n}$ starts at a collision with a fresh atom with $p^1_{A,n}$ starts at a collision with a fresh atom with $p^1 = -v_1$, it follows from (3.22) and (3.23) that

$$\begin{aligned} &\{j = j_0\} \cap \{j(\mathrm{c})\, j+k \text{ for some } k \geqslant 3\} \cap \{p^1_{(j)} = v_1\} \\ &\quad \subseteq \{j = j_0\} \cap \{\text{the stick collides with a fresh atom with } p^1 = v_1 \\ &\qquad\qquad \text{at some time } s \in \Delta^-_{A,j} \text{ and encounters at least } k \\ &\qquad\qquad \text{fresh atoms with } p^1 = -v_1 \text{ during } [s, s+u] \text{ and} \\ &\qquad\qquad \text{(3.22) and (3.23) hold}\} \end{aligned} \tag{3.24}$$

(ii) The precollision velocity of the atom before the first collision is $p^1_{(j)} = -v_1$. Then the atom in all the subsequent collisions is to the right of the stick. Again, we have the following bound for the postcollision velocity of the atom:

$$p^{-1}_{(j)} \leqslant -v_1 + f\tau^-_{A,j} \tag{3.25}$$

Let s be the time of the first collision of the atom. Then to have a recollision at time u

$$\int_s^{s+u} V(t)\, dt = \bar{p}^1_j u \tag{3.26}$$

and by (3.25)

$$\int_s^{s+u} [V(t) + v_1]\, dt \leqslant f\tau^-_{A,j} u, \qquad u \leqslant l/v_2 \tag{3.27}$$

which is the same as before. Hence

$$\begin{aligned} &\{j = j_0\} \cap \{j(\mathrm{c})\, j+k \text{ for some } k \geqslant 3\} \cap \{p^1_{(j)} = -v_1\} \\ &\quad \subseteq \text{right side of (3.24)} \end{aligned} \tag{3.28}$$

We shall denote by $E(\tau^-_{A,j})$ the event on the right of the inclusions (3.24), (3.28). Hence

$$P(\{j = j_0\} \cap \{j(\mathrm{c})\, j+k \text{ for some } k \geqslant 3\}) \leqslant P(E(\tau^-_{A,j})) \tag{3.29}$$

Let $B(\tau^-_{A,j})$ denote the event

$$\begin{aligned} B(\tau^-_{A,j}) = \{&\text{during } \Delta^-_{A,j} \text{ the stick collides with more than two} \\ &\text{fresh atoms with } p^1 = -v_1\} \end{aligned} \tag{3.30}$$

Then

$$G_A^\alpha \cap \{j = j_0\} \cap B(\tau_{A,j}^-)$$
$$\subseteq \{\text{at least three collisions with fresh atoms with } p^1 = -v_1 \text{ occur during } [S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 5A^{-1+\alpha}]\} \cap G_A^\alpha \cap \{j = j_0\} \tag{3.31}$$

This may be seen by considering the complement of $B(\tau_{A,j}^-)$ on $G_A^\alpha \cap \{j = j_0\}$, i.e.,

$$\{\text{the stick has less than three collisions with fresh atoms with } p^1 = -v_1 \text{ during } [S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 5A^{-1+\alpha}]\}$$
$$\subseteq \{\text{during } \Delta_{A,j}^- \text{ the stick collides with less than three fresh atoms with } p^1 = -v_1\} \tag{3.32}$$

Now, on the event on the left we must have at least three collisions with fresh atoms with $p^1 = v_1$ during $[S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 5A^{-1+\alpha}]$. But then

$$\Delta_{A,j}^- \subseteq [S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 5A^{-1+\alpha}]$$

and the inclusion follows.

Note that $S_{k_{2(j-1)}}$ [cf. (3.1) and (3.4)] is a Markov time, i.e., the left collisions with fresh atoms after the time $S_{k_{2(j-1)}}$ are governed by the rate R_A^-. Note that on $\{j = j_0\}$ (recall our key observation)

$$\sup_{\substack{t \in [0, 5A^{-1+\alpha}] \\ V_A(S_{k_{2(j-1)}} + t) < 0}} R_A^-(S_{k_{2(j-1)}} + t) \leqslant \rho l f \, 5A^{-1+\alpha}$$

Hence

$$P(\{j = j_0\} \cap \{\text{at least three collisions with atoms with } p^1 = -v_1 \text{ occur during } [S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 5A^{-1+\alpha}]\}$$
$$\leqslant 5\rho l f A^{-1+\alpha} - P(\{\text{there exist at least three points in } [0, 5A^{-1+\alpha}]\})$$
$$\leqslant [\rho l f (5A^{-1+\alpha})^2/2]^3/3! = D_1 A^{-3/2+6\alpha}$$

where we have used in the second inequality the strong Markov property and the "Poisson domination" (cf. Lemma 3.1).

We use (3.32) now in (3.29) as follows:

$$P(E\tau_{A,j}^-) \leqslant D_1 A^{-3/2+6\alpha} + P((E\tau_{A,j}^-) \cap B(\tau_{A,j}^-)^c) \tag{3.33}$$

By simply counting the numbers of fresh collisions, we obtain that on $B(\tau_{\bar{A},j})^c \cap \{j = j_0\}$

$$\tau_{A,j} \leqslant 5A^{-1+\alpha} \tag{3.34}$$

and for (3.22) we obtain

$$\int_s^{s+u} [V(t) + v_1]\, dt \leqslant f5A^{-1+\alpha}u \tag{3.35}$$

By the strong Markov property and observing that the right of (3.35) multiplied by $\frac{1}{2}\rho l$ is a bound on the collision intensity during $[s, s+u]$, we may simply estimate by (3.23)

$$\begin{aligned} &P(E(_{\bar{A},j})\, B(_{\bar{A},j})^c) \\ &\quad \leqslant \rho l f 5A^{-1+\alpha} - P(\text{at least } k \text{ points in } (0, l/v_2)) \\ &\quad \leqslant D_2 A^{k(-1/2+\alpha)} \end{aligned} \tag{3.36}$$

For $k \geqslant 3$ and A large enough, (3.33) and (3.36) yield

$$P(\{j = j_0\} \cap \{j(\mathrm{c})(j+k) \text{ for some } k \geqslant 3\}) \leqslant A^{k(-1/2+\alpha)} \tag{3.37}$$

Hence, for $k = 3$, we may choose $\beta = 1/2 - 3\alpha$.

Proof of Step 2. For simplicity we start with

(i) One atom connects all the intervals, i.e., some fresh atom with $p^1 = \pm v_1$ collides with the stick during the interval $\Delta_{\bar{A},j}$ and recollides during $\Delta_{\bar{A},j+n}$ and $\Delta_{\bar{A},j+k}$, $n = 1, 2, \quad k \geqslant 3$ (3.38)

We observe that in recollisions the atom in question is pushed further away from the stick. Hence the estimate os step 1 essentially applies and thus the right-hand side of (3.37) is an upper bound for the probability of (3.38).

(ii) The situation is different if more than one fresh atom is connecting. Suppose first that another fresh atom connects $j+n$ with $j+k$, $n = 1, 2$; $k \geqslant 3$. Since the postcollision velocities of the stick due to recollisions during $\Delta_{\bar{A},j}$ cannot become larger than $-v_1 + f\tau_{\bar{A},j}$, the postcollision velocity $\bar{p}'^1$ of the fresh atom satisfies in both the cases $(p'^1 = \pm v_1)$ the inequality:

$$\bar{p}'^1_{(j+n)} \leqslant -v_1 + f\tau_{\bar{A},j} + f \sum_{h=1}^{n} \tau_{\bar{A},j+h}, \qquad n = 1, 2 \tag{3.39}$$

On the set $\{j = j_0\} \cap \{j(\mathrm{c})(j+1)$, $(j+1)(\mathrm{c})(j+2)$, and $(j+2)(\mathrm{c})(j+k)$ for some $k \geqslant 3\}$ the connection between $(j+1)$, $(j+2)$ and $(j+2)$, $(j+k)$ could be due to one or two fresh atoms.

For this, note that (3.39) serves also as a bound for all postcollision velocities of the atoms that collided with the stick during $\Delta_{A,j}^{-}$, n = 1, 2. Hence the bound (3.39) replaces the bounds (3.21) and (3.25) of step 1. Furthermore, it is easily seen that the probabilities of different possibilities of connections can be estimated by adequate combinations of step 1, observing, however, (3.39). Considering also (3.37), it is rather straightforwardly seen that the above event has probability less than $C_2 A^{-3/2+3\alpha}$. ∎

A consequence of Lemma 3.1 is the following.

Lemma 3.3. For all $0 < j \leqslant N_A(T)$, $n_0 > 1$, and any $\gamma > \alpha$

$$P(\{\tau_{A,j}^{-} > A^{-1+\gamma}\} \cap G_A^{\alpha} \cap \Omega_A^{-}(T)) \leqslant C_4 A^{2\alpha n_0 - n_0/2} \tag{3.40}$$

for A large enough.

Proof. By Lemma 3.1, on $\Omega_A^{-}(T)$, j cannot be connected with any $h < j-2$. Let

$$B(\tau_{A,k}^{-}, n^0) = \{\text{during } \Delta_{A,k}^{-} \text{ the stick collides with more than } (n_0 - 1) \text{ fresh atoms with } p^1 = -v_1\}$$

Then on $G_A^{\alpha} \cap ([\bigcap_{k=0}^{2} B(\tau_{A,j-k}^{-}, n_0)^c])$ and for A large enough, $\tau_{A,j}^{-}$ is certainly smaller than $\Delta^{-1+\gamma}$ for $\gamma > \alpha$, which easily follows by noting that on $G_A^{\alpha} \cap ([\bigcap_{k=0}^{2} B(\tau_{A,j-k}^{-}, n_0)^c])$ and during $\Delta_{A,j}^{-}$ the stick collides at most with $n_0 - 1$ fresh atoms with $p^1 = -v_1$ and can recollide at most with $6n_0 + 2$ atoms which collided first during $\Delta_{A,j-1}^{-}$ and $\Delta_{A,j+2}^{-}$ [see definition of $\Delta_{A,j}^{-}$ in (3.8)]. Hence

$$\begin{aligned}
&P(\{\tau_{A,j}^{-} > A^{-1+\gamma}\} \cap G_A^{\alpha} \cap \Omega_A^{-}(T)) \\
&\quad \leqslant P(\{B(\tau_{A,j-1}^{-}, n_0) \cap \{(j-2) \\
&\qquad \text{is not collected with any } h < (j-2)\} \cap G_A^{\alpha}) \\
&\qquad + P(\{B(\tau_{A,j-2}^{-}, n_0)^c \cap B(\tau_{A,j-1}^{-}, n_0) \cap \{(j-2) \\
&\qquad\quad \text{is not collected with any } h < (j-2)\} \cap G_A^{\alpha}) \\
&\qquad + P(B(\tau_{A,j-2}^{-}, n_0)^c \cap B(\tau_{A,j-1}^{-}, n_0)^c \cap B(\tau_{A,j}^{-}, n_0) \cap G_A^{\alpha} \cap \Omega_A^{-}(T))
\end{aligned} \tag{3.41}$$

Since all three terms on the rhs of (3.41) can be dealt with in the same way, we only estimate the last one.

By similar arguments used to prove (3.32), we have that

$$B(\tau^-_{A,j-2}, n_0)^c \cap B(\tau^-_{A,j-1}, n_0)^c \cap B(\tau^-_{A,j}, n_0) \cap G^\alpha_A \cap \Omega^-_A(T)$$
$$\subseteq \{\text{at least } n_0 \text{ collisions with fresh atoms with } p^1 = -v_1 \text{ occur during } [S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + (6n_0+3)A^{-1+\alpha}]\}$$

and then

$$P(B(\tau^-_{A,j-2}, n_0)^c \cap B(\tau^-_{A,j-1}, n_0)^c \cap B(\tau^-_{A,j}, n_0) \cap G^\alpha_A \cap \Omega^-_A(T))$$
$$\leqslant [\rho lf(6n_0+3)^2 A^{-2+2\alpha}/2]^{n_0}/n_0! = D_5 A^{-(1/2-2\alpha)n_0} \quad \blacksquare \tag{3.42}$$

For the following definition we suppose that at $S_{k_{2j-1}}$ the stick is surrounded by an equilibrium distribution of fresh atoms; when the stick now moves to the right, this situation changes, since its velocity becomes larger than v_1 and thus creates a shadow with a nonequilibrium density of fresh atoms (see Fig. 1). Increments for which collisions happen with these "shadow atoms" are dealt with in the next lemma.

For all $h \geqslant 0$, for which during $\Delta_{A,j+h}$ the stick did not yet cross the line

$$Q(S_{k_{2j-1}}) + v_1\lambda, \qquad \lambda \geqslant 0 \tag{3.43}$$

we define $\hat{t}_{j+(h)}$ to be the smaller of the two solutions of

$$Q^+_{A,j} + \Theta_1(h) \sum_{n=1}^{h} Q_{A,j+n} - v_1 \hat{t}_{j+(h)} + \frac{1}{2} f\hat{t}^2_{j+(h)}$$
$$= v_1 \left(\tau^+_{A,j} + \Theta_1(h) \sum_{n=1}^{h} \tau_{A,j+(n)} + \hat{t}_{j+(h)} \right) \tag{3.44}$$

or $\hat{t}_{j+(h)} = \infty$ otherwise, Θ_1 being the Heaviside function at 1. Note that $t_{j+(h)}$ is the time (from $S_{k_{2(j+h)}}$ on) for the stick to leave its shadow (created from $S_{k_{2j-1}}$) if no further collisions with shadow atoms occur. In abuse of notation we thus associate with each $S_{k_{2n}}$ a time $\hat{t}_n$, n standing for $j+h$ and $j+(h)$, respectively, and $S_{k_{2j-1}}$ is the time for which the definition above holds.

Definition 3.2. Let $\Omega^+_A(T)$ be the set of trajectories for which there does not exist j such that for some $h \geqslant 2$ and for all $n = 0, 1, 2,$

$$S_{k_{2(j+n)}+1} \in [S_{k_{2(j+1)}}, S_{k_{2(j+1)}} + \hat{t}_{j+n}] \tag{3.45}$$

$\Omega^+_A(T)$ is the set of trajectories for which no more than two subsequent collisions with "shadow" atoms occur. Set

$$\Omega_A(T) = \Omega^-_A(T) \cap \Omega^+_A(T) \cap G^\alpha_A$$

Lemma 3.4. For all $T < +\infty$

$$P(\Omega_A(T)^c) \leqslant C_5 A^{-\beta}$$

Proof. Let j_1 be the first index for which (3.45) holds; then

$$\Omega_A^+(T)^c = \bigcup_{j=0}^{N_A(T)-3} \{j = j_1\} \tag{3.46}$$

Observe that

$$P(\Omega_A(T)^c) \leqslant P(\Omega_A^+(T)^c \cap [\Omega_A^-(T) \cap G_A^\alpha]) + P([\Omega_A^-(T) \cap G_A^\alpha]^c)$$

Hence, in view of Lemma 3.1, Remark 3.1, and Lemma 3.2, we need only to show that

$$P(\{j = j_1\} \cap \Omega_A^-(T)) \leqslant D_6 A^{-3/2+\beta} \tag{3.47}$$

From (3.44) we easily get

$$\hat{t}_{j+n} \leqslant \frac{f}{2v_1} \left(\sum_{h=0}^{n} \tau_{A,j+h}^+ \right)^2 \tag{3.48}$$

First observe that

$$\{j = j_1\} \subseteq \Bigg\{ \text{the stick collides with a fresh atom with } p^1 = v_1 \text{ during } \left[S_{k_{2j}}, S_{k_{2j}} + \frac{f}{2v_1} (\tau_{A,j}^+)^2 \right], \text{ with a different fresh atom with } p^1 = v_1 \text{ during } \left[S_{k_{2(j+1)}}, S_{k_{2(j+1)}} + \frac{1}{2v_1} (\tau_{A,j}^+ + \tau_{A,j+1}^+)^2 \right] \text{ and again with a different fresh atom with } p^1 = v_1 \text{ during } \left[S_{k_{2(j+2)}}, S_{k_{2(j+2)}} + \frac{f}{2v_1} \left(\sum_{h=0}^{2} \tau_{A,j+h}^+ \right)^2 \right] \Bigg\} \tag{3.49}$$

Suppose now that for any $j = 0, \ldots, N_A(T)$ and $\gamma > \alpha$

$$P(\{\tau_{A,j}^+ > 2A^{-1+\gamma}\} \cap \Omega_A^-(T)) \leqslant C_6 A^{-3/2+6\alpha} \tag{3.50}$$

Then (3.47) easily follows by "Poisson domination" with the $\sigma l(2v_1 + 2fA^{-1+\gamma}) - P$ law. So we prove (3.50). First, note that

$$\{\tau_{A,j}^+ > 2A^{-1+\gamma}\} \subseteq \{\text{the stick collides with a fresh atom with } p^1 = -v_1 \text{ during } [S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 2A^{-1+\gamma}]\} \equiv \mathscr{A} \tag{3.51}$$

which easily follows by taking complements and by observing (3.4). But by (3.40) with $n_0 = 3$ and $1/12 > \alpha > 0$, and Remark 3.1,

$$\begin{aligned}
&P(\mathscr{A} \cap \Omega_A^-(T)) \\
&\quad \leqslant P(\{\tau_{A,j}^- > A^{-1+\gamma}\} \cap \Omega_A^-(T) \cap G_A^\alpha) + P(G_A^{\alpha c}) \\
&\qquad + P(\mathscr{A} \cap \{\tau_{A,j}^- < A^{-1+\gamma}\}) \\
&\quad \leqslant C_4 A^{-3/2+6\alpha} + C_6 \exp(-\theta^{-1} A^\alpha) \\
&\qquad + P(\{\mathscr{A} \cap \{\tau_{A,j}^- < A^{-1+\gamma}\}\})
\end{aligned} \tag{3.52}$$

Note that on $\mathscr{A}$, $[S_{k_{2(j-1)}}, S_{k_{2(j-1)}} + 2A^{-1+\gamma}]$ is contained in $\Delta_{A,j}$. Therefore, if $\tau_{A,j}^- < A^{-1+\gamma}$ we have that $V_A > 0$ at least during $[S_{k_{2(j-1)}} + A^{-1+\gamma}, S_{k_{2(j-1)}} + 2A^{-1+\gamma}]$. After $S_{\bar{k}_{2j-1}}$, V_A can only become negative through a collision with a fresh atom with $p^1 = -v_1$, which is precluded on $\mathscr{A}$. Hence

$$\begin{aligned}
&P(\{\tau_{A,j}^- < A^{-1+\gamma}\} \cap \mathscr{A}) \\
&\quad \leqslant P(\{\text{there is no collision with a fresh atom with } p^1 = -v_1 \\
&\qquad\quad \text{during } [S_{k_{2(j-1)}} + A^{-1+\gamma}, S_{k_{2(j-1)}} + 2A^{-1+\gamma}] \text{ given } V_A(t) \\
&\qquad\quad \text{is positive for all } t \in [S_{k_{2(j-1)}} + A^{-1+\gamma}, S_{k_{2(j-1)}} + 2A^{-1+\gamma}]\}) \\
&\quad \leqslant \rho l v_1 - P(\{\text{no points in } [0, A^{-1+\gamma}]\}) \leqslant \exp(-\theta^1 A^\gamma)
\end{aligned} \tag{3.53}$$

by the obvious "Poisson domination." ∎

We find it useful to redefine the sequence $\{Q_{A,j}^\pm, \Delta_{A,j}^\pm\}$, $j = 0,..., N_A(T)$. Suppose that j is the index of the first increment for which $\Delta_{A,j}^-$ ends in a collision with a "shadow" atom; then we wish to consider $(Q'^+_{A,j-1}, \Delta'^+_{A,j-1})$, $Q'^+_{A,j-1} = Q^+_{A,j-1} + Q_{A,j}$, $\Delta'^+_{A,j-1} = \Delta^+_{A,j-1} \cup \Delta_{A,j}$, as one positive (increment, interval) if $\Delta^-_{A,j+1}$ does not end in a collision with a "shadow" atom; otherwise, $Q'^+_{A,j-1} = Q^+_{A,j-1} + Q_{A,j} + Q_{A,j+1}$ and $\Delta'^+_{A,j-1} = \Delta^+_{A,j-1} \cup \Delta_{A,j} \cup \Delta_{A,j+1}$. We repeat this for all following increments (see Fig. 1).

This construction takes care of the trajectories in $\Omega_A(T)$; on $\Omega_A(T)^c$ we leave the increments as they are. Let us denote by $\{(Q'^\pm_{A,j}, \Delta'^\pm_{A,j})\}_{j=0,..., N'_A(T)}$ the new sequence.

Since for each $\Delta'^-_{A,j}$ there exists some $\Delta^-_{A,n}$ we shall from now on denote by $S'_{k_{2(j-1)}}$, $S'_{k_{2j-1}}$ and $S'_{\bar{k}_{2j-1}}$, $S_{k_{2(n-1)}}$, $S_{k_{2n-1}}$, and $S_{\bar{k}_{2n-1}}$, respectively.

We are now in a position to define "good" (the generic) increments and "bad" increments.

Definition 3.3. We shall call an increment $Q'_{A,j}$ "generic" if:

(i) There exists n such that either $Q'^+_{A,j} = Q^+_{A,n}$ or $Q'^+_{A,j} = Q^+_{A,n} \cup Q_{A,n+1}$ and if one of the following three conditions hold:

(ii) $k_{2j-1}-k_{2(j-1)}=0$ (then the stick has no recollision during $\Delta'^{-}_{A,j}$).

(iii) $k_{2j-1}-k_{2(j-1)}=1$ and during the interval $\Delta'^{-}_{A,j}$, after the time $S'_{k_{2j-1}+1}$, the stick has no recollision with atoms that collided first during some interval $\Delta'_{A,k}$ with $k<j$.

(iv) $k_{2j-1}-k_{2(j-1)}>1$, and the stick always collides with the same atom until $S'_{k_{2j-1}}$, and the next collision where the stick gets again a negative velocity is a collision with a fresh atom.

Let

$$N_1=\{j: 0\leqslant j\leqslant N'_A(T) \text{ and } Q'_{A,j} \text{ is a "generic" increment}\} \tag{3.54}$$

and

$$N_2=\{j: 0\leqslant j\leqslant N'_A(T) \text{and } j\notin N_1\} \tag{3.55}$$

The lemma we shall prove next using Lemma 3.3 justifies the terminology.

Lemma 3.5. Let N_2 be the set defined in (3.55). Then

$$P(\Omega_A(T)\cap\{j\in N_2\})\leqslant C_7 A^{-1+2\gamma}$$

Proof. By definition of the set N_2 it follows that

$$\begin{aligned}\{j\in N_2\}\subseteq&\{\exists k \text{ such that } Q'^{+}_{A,j}=Q^{+}_{A,k}+Q_{A,k+1}+Q_{A,k+2}\}\\ &\cup(\{\exists k\colon Q'^{+}_{A,j}=Q^{+}_{A,k} \text{ or } Q'^{+}_{A,j}=Q^{+}_{A,k}+Q_{A,k+1}\}\\ &\cap\{\text{one of the conditions (ii), (iii), (iv) does not hold}\})\end{aligned} \tag{3.56}$$

From (3.48) it easily follows, as in (3.49), that

$$P(\{\exists k \text{ such that } Q'^{+}_{A,j}=Q^{+}_{A,k}+Q_{A,k+1}+Q_{A,k+2}\}\cap\Omega_A(T))\leqslant D_7 A^{-1+4\alpha} \tag{3.57}$$

It only remains to estimate the probability of the second set on the rhs of (3.56). If j does not satisfy one of the condition (iii), (iv), then one of the following occurs:

(a) $k_{2j-1}-k_{2j-2}>1$ and the stick collides with at least two different atoms during $\Delta'^{-}_{A,j}$ and before the time $S'_{k_{2j-1}+1}$,

(b) $k_{2j-1}-k_{2j-2}=1$ and the stick recollides after the time $S'_{k_{2j-1}+1}$ during $\Delta'^{-}_{A,j}$.

On $\Omega_A(T)\cap\{\exists k\colon Q'^{+}_{A,j}=Q'^{+}_{A,k}$ or $Q'^{+}_{A,j}=Q^{+}_{A,k}+Q_{A,k+1}\}$ the increment $Q'^{-}_{A,j}$ can always be identified with some increment $Q^{-}_{A,k(j)}$ for some

$0 \leqslant k(j) \leqslant N_A(T)$, and $Q^-_{A,k(j)}$ can only be connected with $Q^-_{A,k(j)-1}$ and $Q^-_{A,k(j)-2}$.

Suppose, first, that $Q^-_{A,k(j)}$ is not connected with any $Q_{A,n}$ for $n<k$. If (a) and (b) occur for $Q'^-_{A,j}$, then

$$\begin{aligned}
&\{j \in N_2\} \cap \{k(j) \text{ is not connected with any } n < k(j)\}\\
&\quad \subseteq \{k(j) \text{ is not connected with any } n < k(j)\}\\
&\qquad \cap (\{\text{at least two fresh atoms with } p^1 = -v_1 \text{ collide with the stick during } [S'_{k_{2(j-1)}}, S'_{k_{2j-1}}]\}\\
&\qquad \cup \{\text{the collides with one fresh atom with } p^1 = -v_1 \text{ during } (S'_{k_{2(j-1)}}, S'_{k_{2j-1}}] \text{ and with at least one fresh atom with } p^1 = -v_1\\
&\qquad\quad \text{at some time } s > S'_{k_{2j-1}+1}\})
\end{aligned} \tag{3.58}$$

Note that for $t \in [S'_{k_{2(j-1)}}, S'_{k_{2j-1}}]$, $R^-_A(t) \leqslant \rho lf(t - S'_{k_{2(j-1)}})/2$ and $R^+_A(t) \geqslant \rho l v_1$.

Then we use "Poisson domination" with $\rho lf(S'_{k_{2j-1}} - S'_{k_{2(j-1)}}) - P$ law for the collision event and "Poisson domination" with $\rho l v_1 - P$ law to control the length of the interval $S'_{k_{2j-1}} - S'_{k_{2(j-1)}}$; hence we obtain that

$$\begin{aligned}
&P(\{j \in N_2\} \cap \{k(j) \text{ is not connected with any } n < k(j)\})\\
&\quad \leqslant \frac{1}{2}\left[\int_0^\infty dt \left(\rho \frac{fl}{2}\right)^2 e^{-\rho l v_1 t}\right] + \int_0^\infty dt \frac{\rho lf}{2} t e^{-\rho l v_1 t} \int_0^\infty ds \frac{\rho lf}{2}(t+s)\, e^{-\rho l v_1 s}\\
&\quad \leqslant D_8 A^{-1}
\end{aligned} \tag{3.59}$$

Suppose now that $k(j)$ is connected with $k(j)-1$, but $(k(j)-2)$, $(k(j)-1)$, and $k(j)$ are not connected. From (3.59) we obtain

$$\begin{aligned}
&P(\{k_{2k(j)-1} - k_{2(k(j)-1)} \geqslant 2\}\\
&\quad \cap \{k(j) \text{ is not connected with any } n < k(j)-1\})\\
&\quad \leqslant D_8 A^{-1}
\end{aligned} \tag{3.60}$$

and similarly, observing also (3.37),

$$\begin{aligned}
&P(\{k_{2k(j)-1} - k_{2(k(j)-1)} = 1\}\\
&\quad \cap \{k(j)-1 \text{ is not connected with any } n < k(j)-1\}\\
&\quad \cap \{(k(j)-1)(\mathrm{c})\, k(j)\})\\
&\quad \leqslant D_9 A^{-1+2\alpha}
\end{aligned} \tag{3.61}$$

By virtue of (3.60) and (3.61) we have only to deal with $k_{2k(j)-1} - k_{2(k(j)-1)} = 0$ and $(k(j)-1)(c)\, k(j)$, i.e., no fresh atom with

$p^1 = -v_1$ collides with the stick during $\Delta^-_{A,k(j)-1}$ and one fresh atom with $p^1 = v_1$ collides first during $\Delta^-_{A,k(j)}$. That is, a recollision and at least one fresh collision occur during $\Delta^-_{A,k(j)}$. Similarly as in (3.59), we have that

$$P(\{j \in N_2\} \cap \{k_{2k(j)-1} - k_{2(k(j)-1)} = 0\} \cap \{k(j) - 1(\mathrm{c})\, k(j)\}) \leqslant D_{10} A^{-1} \tag{3.62a}$$

$$\begin{aligned} &\int_0^\infty dt\, \rho l v_1 e^{\rho l v_1 t} \int_t^{t + flt/2v_1 v_2} ds\, \rho l \left[v_1 + \frac{f}{2}(t+s) \right] e^{-\rho l v_1 (s-t)} \\ &\quad \times \int_0^\infty du \frac{\rho l f}{2} (t+u)\, e^{-\rho l v_1 u} \\ &\quad \leqslant D_{10} A^{-1} \end{aligned} \tag{3.62b}$$

The first two integrals in inequality (3.62b) are bounds for the probability that the stick collides with a fresh atom with $p^1 = v_1$ during $\Delta^-_{A,k(j)-1}$ and recollides with it during $\Delta^-_{A,k(j)}$; observing (3.23) and (3.24), the third integral is a bound for the probability that the stick collides with at least one fresh atom with $p^1 = -v_1$ during $\Delta^-_{A,k(j)}$.

To conclude the proof of the lemma, we remark that there exists $\beta < 1/2$ such that

$$P(\{k(j), k(j)-1, k(j)-2 \text{ are connected}\} \cap \Omega_A(T) \leqslant C_8 A^{-1+\beta} \tag{3.63}$$

which follows from the proof of Lemma 3.1. ∎

Remark. An increment is roughly of the order of $A^{-1/2}$. A bad increment, i.e., an element of N_2, appears with probability $A^{-1+2\gamma}$, and thus contributes to the process with order $A^{-3/2+2\gamma}$ and if there total is A, their total effect vanishes like $A^{-1/2+2\gamma}$. Thus, the process will be determined in the limit by generic increments, which are all pictured in Fig. 1. We make this precise in the next section.

4. AN APPROXIMATION

We shall now define a process $\hat{Q}_A = \{\hat{Q}_A(t)\}_{t \in [0,T]}$ which is essentially built from the generic increments of Q_A. These are pictured in Fig. 1. To understand the construction of the typical increment of $\hat{Q}_A$, recall that the stick has in the generic increment the possibility of recolliding. The recollision is captured in the description of X_A below. These define in sense $\hat{Q}_A$ abstractly. Then we couple $\hat{Q}_A$ and Q_A, i.e., we realize them simultaneously on the same space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$, on which we shall show that they are close in probability as $A \to \infty$. We shall construct $\hat{Q}_A$ on the probability space

$$(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}) = (\Omega, \mathscr{F}, P) \times (\mathscr{H}, \mathscr{A}, \mathscr{P})$$

where $(\mathscr{H}, \mathscr{A}, \mathscr{P})$ is a suitable probability space to be specified later. The realization of Q_A on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ is obvious.

To construct $\hat{Q}_A$, we introduce a virtual stick, whose motion we shall describe next, defining a typical increment X_A. The virtual stick is invisible for the atoms, and we use the terms crossings, encounters, etc., when an atom passes through the virtual stick.

Put the virtual stick at the origin of the Poisson bath. The virtual stick starts with velocity $V = -v_1$ and it moves according to Newton's law in the force field F_A. Atoms which crossed the virtual stick are taken out of the bath; the stick continues its motion unaltered. Let

$S_A =$ time of the first crossing of the virtual stick by an atom with $p^1 = v_1$

if no atom with $p^1 = -v_1$ reached the virtual stick before.

If the virtual stick encounters an atom $p^1 = -v_1$, $p^2 = -v_2$ $(+v_2)$, first, say, at y (parametrization of the length of the stick) at time $\bar{s}$, then let

$$S_A = \begin{cases} \text{the time of the first crossing after } \bar{s} \text{ by an atom} \\ \text{with } p^1 = v_1 \text{ if this time is larger than } \bar{s} + y/v_2 \ [\bar{s} + (l-y)/v_2] \\ \text{the time of the second crossing after } \bar{s} \text{ by an atom with} \\ p^1 = v_1, \text{ otherwise ("recollision")} \end{cases}$$

We define

$$t_A^- = \min(S_A, 2v_1/f), \qquad X_A^- = -v_1 t_A^- + \tfrac{1}{2} f (t_A^-)^2 \tag{4.1}$$

Now, we put the virtual stick at the point $-v_1 t_A^- + \frac{1}{2} f (t_A^-)^2$ at time t_A^-. The virtual stick starts with velocity $V = +v_1$.

The stick moves again according to Newton's law in the force field $\mathbf{F}_A$. Atoms which cross are taken out of the bath. Let $t_A^- + \tilde{t}_A^+$ be the time of the first crossing of the virtual stick by an atom with velocity $p^1 = -v_1$ and let $\hat{s}$ be the smaller of the two solutions of

$$v_1 \tilde{t}_A^+ + \tfrac{1}{2} f \tilde{t}_A^{+2} - v_1 \hat{s} + \tfrac{1}{2} f \hat{s}^2 = v_1 (\tilde{t}_A^+ + \hat{s}) \tag{4.2}$$

or $\hat{s} = \infty$ otherwise. Now the virtual stick splits into two: One part changes its velocity to $-v_1$ and moves under the force until $t_A^- + t_A^+ \equiv t_A^- + \tilde{t}_A^+ + f(\tilde{t}_A^+)^2/4v_1$, if it does not encounter an atom with $p^1 = v_1$ during $[t_A^- + \tilde{t}_A^+, t_A^- + \tilde{t}_A^+ + \bar{s}]$, and if it does, we follow the undisturbed motion of the second stick until $t_A^- + t_A^+ \equiv t_A^- + \tilde{t}_A^+ +$ (the time until the second crossing by an atom with $p^1 = -v_1$).

Then we set

$$X_A^+ = v_1 t_A^+, \qquad X_A = X_A^- + X_A^+, \qquad t_A = t_A^- + t_A^+$$

We now define the increments $\hat{Q}_{A,j}$ for a given trajectory of the mechanical stick. $\hat{Q}'_{A,j}$ is obtained by putting the virtual stick at the point $Q_A(S'_{k2(j-1)}) - v_1 t_{j-1} + \frac{1}{2} f \hat{t}'_{j-1}$, where $\hat{t}'_{j-1}$ is the time defined in (3.44). Note that $Q_A(S'_{k2(j-1)}) - v_1 \hat{t}'_{j-1} + \frac{1}{2} f(\hat{t}'_{j-1})$ is the position of the mechanical stick at time $S'_{k2(j-1)} + \hat{t}'_{j-1}$ in case of no collisions with "shadow" atoms. From this time on, the mechanical stick "sees" again fresh atoms with the equilibrium rate.

The virtual stick behaves now according to the above description of the typical increment X_A where the time-zero Poisson bath is replaced by the bath seen by the virtual stick at time $S'_{k2(j-1)} + \hat{t}'_{j-1}$ and accordingly we shift the times $\bar{s}$, S_A, and t_A^- by that amount to $\bar{s}_j$, $S_{A,j}$, and $t_{A,j}^-$ and we deote the times corresponding to $\tilde{t}_A^+$ and $\hat{s}$ by $\tilde{\tau}_{A,j}^+$ and $\hat{s}_j$. Then we set

$$\hat{\tau}'^-_{A,j} = \min(S_{A,j}, 2v_1/f), \qquad \text{and} \qquad \hat{Q}'^-_{A,j} = -v_1 \hat{\tau}'^-_{A,j} + \frac{1}{2} f \hat{\tau}'^{-2}_{A,j} \tag{4.3}$$

and

$$\hat{\tau}'^+_{A,j} \equiv \begin{cases} \tilde{\tau}^+_{A,j} + \dfrac{f}{4v^1} \tilde{\tau}^{+2}_{A,j} & \text{if there is no crossing by a fresh atom with } p^1 = v_1 \text{ during } [S'_{k2(j-1)} + \hat{t}_{j-1} + \tilde{\tau}^+_{A,j}, S'_{k2(j-1)} + \hat{t}_{j-1} + \tilde{\tau}^+_{A,j} + \hat{s}_j] \\ \tilde{\tau}^+_{A,j} + (\text{the time until the second crossing by an atom with } p^1 = -v_1) & \text{otherwise} \end{cases} \tag{4.4}$$

Furthermore, let

$$\hat{Q}'^+_{A,j} = v^1 \hat{\tau}'^+_{A,j}, \qquad \hat{Q}'_{A,j} = \hat{Q}'^-_{A,j} + \hat{Q}'^+_{A,j}$$

Note that $\hat{Q}'_{A,j}$ is a version of X_A.

We say that $\hat{Q}'_{A,j}$ has *property* $\bar{E}$ is all atoms which define $\hat{Q}'_{A,j}$ collide with the mechanical stick during $\Delta'_{A,j}$. Then we define

$$\hat{Q}_{A,j} = \begin{cases} \hat{Q}'_{A,j} & \text{if it has property } \bar{E} \quad (\text{"} j \in \bar{E} \text{"}) \\ H_{A,j} & \text{otherwise} \end{cases}$$

where $H_{A,j}$ is a random variable whose distribution is the conditional distribution

$$P_{\hat{Q}'_{A,j}}(\cdot \mid \{\text{"} j \notin \bar{E} \text{"}\})$$

and which is independent of $\hat{Q}'_{A,i}$ and of $H_{A,i}$ for $i \neq j$, and of property $\bar{E}$. We have

$$\hat{\tau}^{\pm}_{A,j} = \begin{cases} \tilde{\tau}'^{\pm}_{A,j} & \text{if } j \text{ has property } \bar{E} \\ \hbar^{\pm}_{A,j} & \text{otherwise} \end{cases} \tag{4.5}$$

where $\hbar^{\pm}_{A,j}$ are random variables whose distributions are the conditional distributions

$$P_{\tilde{\tau}'^{\pm}_{A,j}}(\,|\cdot\{\text{“}j \notin \bar{E}\text{”}\})$$

independent of $\hat{Q}'_{A,i}$ and of $\hbar^{\pm}_{A,i}$ for $i \neq j$ and of property $\bar{E}$.

If $P(\text{“}j \notin \bar{E}\text{”}) = 0$, there $H_{A,j} = 0$ and $\hbar^{\pm}_{A,j} = 0$.

We choose $(\mathscr{H}, \mathscr{A}, \mathscr{P})$ as the space on which we realize the random variables $H_{A,j}$ and $\hbar^{\pm}_{A,j}$. This way, all the $\hat{Q}_{A,j}$ are independent copies of each other. To see this, note that the times $S'_{k_{2(j-1)}} + \hat{t}'_{j-1}$ are such that all the fresh atoms surrounding the mechanical stick are Poisson distributed. Furthermore, due to property E, the atoms defining $\hat{Q}_{A,l}$ do not define $\hat{Q}_{A,k}$ for all $k, l < j$, $l \neq k$. Furthermore, virtual recollisions in $\hat{Q}_{A,j}$, i.e., collisions which are not possible given the increments $\hat{Q}_{A,i}$, $i < j$, do not occur, due to the specific structure of the motion and due to property $\bar{E}$: the “no atom information” during $\Delta^{-}_{A,j}$ moves with $-v_1$ since the atoms defining $\hat{Q}_{A,i}$ also collide during $\Delta'_{A,i}$ and therefore the virtual stick never will be moved to a position where it could catch up to this information. During $\tau^{\pm}_{A,i}$ the “atom information” which could be relevant for the future motion is not registered in the $\hat{Q}^{+}_{A,j}$ increment. Let

$$\hat{T}_A(T) = \sum_{j=0}^{N'_A(T)} \hat{\tau}_{A,j} \tag{4.6}$$

We define

$$\begin{aligned} &\text{(a)} \quad K^{<}_A(t) = \inf\left\{k \leqslant 0 \text{ such that } \hat{T}_A(t) + \sum_{n=0}^{k} t_{A,n} \geqslant t\right\} \quad \text{if} \quad \hat{T}_A(t) < t \\ &\text{(b)} \quad K^{>}_A(t) = \inf\left\{h \geqslant 0 \text{ such that } \sum_{j=0}^{h} \hat{\tau}_{A,j} \geqslant t\right\} \quad \text{if} \quad \hat{T}_A(t) \geqslant t \end{aligned} \tag{4.7}$$

where $t_{A,n} = t^{+}_{A,n} + t^{-}_{A,n}$; $t^{-}_{A,n}$ and $t^{+}_{A,n}$ are independent copies of the random times t^{-}_{A} and t^{+}_{A}. We now define the process $\hat{Q}_A \equiv \{\hat{Q}_A(t)\}_{t \in [0,T]}$:

$$\hat{Q}_A(t) = \begin{cases} \displaystyle\sum_{j=0}^{K^{>}_A(t)-1} \hat{Q}_{A,j} & \text{if} \quad \hat{T}_A(t) \geqslant t \\ \displaystyle\sum_{j=0}^{N'_A(t)} \hat{Q}_{A,j} + \sum_{n=0}^{K^{>}_A(t)-1} X_{A,n} & \text{if} \quad \hat{T}_A(t) < t \end{cases} \tag{4.8}$$

where

$$X_{A,n} = X^+_{A,n} + X^-_{A,n}$$
$$X^+_{A,n} = v^1 t^+_{A,n}, \qquad X^-_{A,n} = -v^1 t^-_{A,n} + \tfrac{1}{2} f t^{-2}_{A,n} \tag{4.9}$$

with $(t^+_{A,n}; t^-_{A,n})$ the random times defined below (4.7).

Lemma 4.1. For all $T < +\infty$, the process $\{Q_A(t)\}_{t \in [0,T]}$ given by (4.8) converges in distribution, as $A \to \infty$, to $\{Q^*(t)\}_{t \in [0,T]}$ on $D([0,T])$ endowed with the Skorohod topology.

Proof. Let

$$K_A(t) = \inf\left\{k\colon \sum_{n=0}^{k} t_{A,n} \geqslant t\right\} \tag{4.10}$$

and let $\{X_A(t)\}_{t \in [0,T]}$ be the following process:

$$X_A(t) = \sum_{n=0}^{K_A(t)-1} [X_{A,n} - \mathbb{E}(X_{A,n})] \tag{4.11}$$

where the $X_{A,n}$ are the random variables defined in (4.9), i.e., independent copies of the random variables X_A.

Then, to show that $\hat{Q}_A$ converges in distribution to Q^* is equivalent to proving that the process $\{X_A(t)\}_{t \in [0,T]}$ converges weakly to a Wiener process with diffusion coefficient $\sigma = (v_1 q)^{1/2}$ and that $\sum_{n=0}^{K_A(t)-1} \mathbb{E}(X_{A,n})$ converges in probability to the deterministic process $\frac{1}{2} f_0 \theta[(1-e^{-r})/r]\, t$.[5]

For this we shall use Corollary 3.8 of McLeish.[6] Let

$$\hat{X}_{A,n} = (X_{A,n} - \mathbb{E}(X_{A,n})) \cdot \chi_{\{n \leqslant K_A(t)\}}$$

and let $\mathscr{F}_{A,n}$ denote the σ-algebra generated by $\{t^+_{A,j}; t^-_{A,j}\}_{j \leqslant n}$. Then $\{X_A(t)\}_{t \in [0,T]}$ converges to a Wiener process with diffusion coefficient $\sigma = (v_1 q)^{1/2}$ if for all $t \in [0,T]$:

$$\text{(a)} \quad \sum_{n=0}^{\infty} \mathbb{E}(\hat{X}^2_{A,n} \cdot \chi_{\{|X_{A,n} - \mathbb{E}(X_{A,n})| > \varepsilon\}} \mid \mathscr{F}_{A,n-1}) \to 0 \text{ in prob. as } A \to \infty$$

$$\text{(b)} \quad \sum_{n=0}^{\infty} \mathbb{E}(\hat{X}^2_{A,n} \mid \mathscr{F}_{A,n-1}) \to v_1 q t \text{ in prob. as } A \to \infty \tag{4.12}$$

$$\text{(c)} \quad \sum_{n=0}^{\infty} |\mathbb{E}(\hat{X}_{A,n} \mid \mathscr{F}_{A,n-1})| \to 0 \text{ in prob. as } A \to \infty$$

Note that

$$\{n \leqslant K_A(t)\} = \left\{ \sum_{j=0}^{n-1} (t_{A,j}^- + t_{A,j}^+) < t \right\} \in \mathscr{F}_{A,n-1}$$

so (c) is obvious.

For the same reason, (a) and (b) may be written as

$$\begin{aligned} &\text{(a)} \quad \sum_{n=0}^{\infty} \chi_{\{n \leqslant K_A(t)\}} \cdot \mathbb{E}((X_{A,n} - \mathbb{E}(X_{A,n}))^2 \cdot \chi_{\{|X_{A,n} - \mathbb{E}(X_{A,n})| > \varepsilon\}}) \to 0 \\ &\qquad \text{in prob. as } A \to \infty \\ &\text{(b)} \quad \sum_{n=0}^{\infty} \chi_{\{n \leqslant K_A(t)\}} \cdot \mathbb{E}((X_{A,n} - \mathbb{E}(X_{A,n}))^2) \to v_1 q t \\ &\qquad \text{in prob. as } A \to \infty \end{aligned} \tag{4.13}$$

By Chebychev's inequalities the left-hand side of (4.13a) is bounded by

$$\begin{aligned} &\sum_{n=0}^{\infty} \chi_{\{n \leqslant K_A(t)\}} \cdot \mathbb{E}([X_{A,n} - \mathbb{E}X_{A,n})]^4)^{1/2} \, P(|X_{A,n} - \mathbb{E}(X_{A,n})| > \varepsilon)^{1/2} \\ &\quad \leqslant \varepsilon^{-1} K_A(t) \, \mathbb{E}([X_{A,0} - \mathbb{E}(X_{A,0})]^4)^{1/2} \, \mathbb{E}([X_{A,0} - \mathbb{E}(X_{A,0})]^2)^{1/2} \end{aligned} \tag{4.14}$$

We compute now $\mathbb{E}(X_A)$, $\mathbb{E}(X_A^2)$, $\mathbb{E}(X_A^4)$, and $\mathbb{E}(t_A)$. We shall start with $\mathbb{E}(t_A) = \mathbb{E}(t_A^+) + \mathbb{E}(t_A^-)$. By definition of the random time t_A^- [Eq. (4.1)] we obtain that

$$\begin{aligned} \mathbb{E}(t_A^-) = &\int_0^{2v_1/f} dt \, \rho l \left(v_1 - \frac{ft}{2} \right) e^{-\rho l v_1 t} t \\ &+ \int_0^l dy \int_0^{2v_1/f} dt \, \rho \frac{1}{2} f t e^{-\rho l v_1 t} \left[\int_t^{t + l/v_2} ds \, \rho l \left(v_1 - \frac{1}{2} f s \right) e^{-\rho l v_1 (u - s)} \right. \\ &\times \int_s^{s + 2v_1/f} du \, \rho l \left(v_1 - \frac{1}{2} f u \right) e^{-\rho l v_1 (u - s)} u \\ &\left. + \int_{t + y/v_2}^{t + y/v_2 + 2v_1/f} du \, \rho l \left(v_1 - \frac{1}{2} f u \right) e^{-\rho l v_1 (u - t)} u \right] \end{aligned} \tag{4.15}$$

Interchanging the integrals, we obtain that

$$\begin{aligned} \mathbb{E}(t_A^-) = &\int_0^{2v_1/f} dt \, \rho l \left(v_1 - \frac{f}{2} t \right) e^{-\rho l v_1 t} t \left[1 + \int_0^{\min(v_2 u, l} dy \frac{1}{4} \rho f \left(t - \frac{y}{v_2} \right)^2 \right. \\ &\left. + \int_0^t ds \frac{1}{2} \rho f s \int_0^l dy \int_s^{\min(t, s + y/v_2)} du \, \rho l \left(v_1 - \frac{1}{2} f u \right) \right] \end{aligned} \tag{4.16}$$

From this we compute

$$\mathbb{E}(t_A^-) = \theta A^{-1} + f_0 \theta^2 v_1^{-1} A^{-3/2} - f_0 \theta^2 v_1^{-1} \left(\frac{1-e^{-r}}{r}\right) A^{-3/2} + O(A^{-2}) \tag{4.17}$$

and the following bounds for $\mathbb{E}((t_A^-)^k)$:

$$\mathbb{E}((t_A^-)^k) = K!\, \theta^k A^{-k} + O(A^{-(k+1/2)}) \tag{4.18}$$

We shall now compute $\mathbb{E}(t_A^+)$. We start with computing the probability of the set [cf. (4.2)]

$$C_A = \{\text{the virtual stick collides with an atom with } p^1 = v_1 \text{ during } [\tilde{t}^+_{A,j}, \tilde{t}^+_{A,j} + \tilde{s}]\}$$

We have

$$P(C_A) = \int_0^\infty dt\, \rho l \left(v_1 + \frac{ft}{2}\right) e^{-\rho l(v_1 t + ft^2/4)} \int_0^{ft^2/2v_1} dP_t(s)$$

$$= \int_0^\infty dt\, \rho l \left(v_1 + \frac{ft}{2}\right) e^{-\rho l v_1 t} \int_{(t - l/v_2) \vee 0}^{t} ds\, v_2 \left(\frac{fs^2}{4}\right) + O(A^{-1}) \tag{4.19}$$

where $P_t(s) = P(\{\text{the stick has a collision with an atom with } p^1 = v_1 \text{ during } [t, t+s]\})$. For this, note that in the corresponding one-dimensional model the probability on the left of (4.19) would be zero, since the accelerated molecule which moves to the right leaves a "shadow" with no atoms in it. When the molecule moves again to the left, then the first time it can collide with an atom moving to the right is only at $\tilde{t}_A^+ + \tilde{s}$. Collisions which are not possible, given the past trajectory, are sometimes called virtual recollisions.

In the two-dimensional model, the stick also has a shadow, but now atoms can enter it (see Fig. 1).

At time s the length of the shadow in one dimension is $fs^2/2$; $\rho v_2(fs^2/2)$ is the rate with which atoms enter the shadow. An atom which entered the shadow leaves it after a time l/v_2; this explains the boundaries of the second integral on the right of (4.19).

From (4.19) we get

$$P(C_A) = \int_0^\infty dt\, \rho l v_1 e^{-\rho l v_1 t} \rho \frac{f v_2}{12} t^3$$

$$+ \int_{l/v_2}^\infty dt\, \rho l v_1 e^{-\rho l v_1 t} \frac{1}{12} f v_2 \left(t - \frac{l}{v_2}\right)^3 + O(A^{-1})$$

$$= \frac{1}{2} f_0 \theta v_1^{-1} \left(\frac{r - e^{-r}}{r}\right) A^{-1/2} + O(A^{-1}) \tag{4.20}$$

By definition of t_A^+ we now obtain that

$$\mathbb{E}(t_{A,0}^+) = \int_0^\infty dt\, \rho l(v_1 + \tfrac{1}{2}ft)\, e^{-\rho l(v_1 t + f^2/4)}(t + ft^2/2v_1)[1 - P_t(ft^2/2v_1)]$$

$$+ \int_0^\infty dt\, \tfrac{1}{2} l\rho(v_1 + \tfrac{1}{2}ft)\, e^{-\rho l(v_1 t + ft^2/4} \int_0^{ft^2/2v_1} dP_t(s) \int_0^\infty du\, \rho l(v_1 + \tfrac{1}{2}fu)$$

$$\times e^{-\rho l(v_1 u + fu^2/4)}[t + u + f(t^2 + s^2 + u^2)/4v_1] \tag{4.21}$$

Thus,

$$\mathbb{E}(t_A^+) = [1 + P(C_A)] \int_0^\infty dt\, l\rho v_1 e^{-\rho l v_1 t} t + O(A^{-5/2})$$

$$= \theta A^{-1} + \frac{1}{2} f_0 \theta^2 v_1^{-1} \left(\frac{1 - e^{-r}}{r}\right) A^{-3/2} + O(A^{-5/2}) \tag{4.22}$$

From (4.21) it follows that for all $k \geqslant 2$.

$$\mathbb{E}((t_A^+)^k) = [1 + P(C_A)] \int_0^\infty dt'\, \rho l v_1 e^{-\rho l v_1 t} t^k + O(A^{-(k+1/2)})$$

$$= [1 + P(C_A)]^k!\, \theta^k A^{-k} + O(A^{-(k+1/2)}) \tag{4.23}$$

From (4.17), (4.18), (4.21), and (4.23) we obtain that

$$\mathbb{E}(t_A) = 2\theta A^{-1} + O(A^{-3/2}) \tag{4.24}$$

and the following estimates for the expectation, the second, and the $(2k)$th moments of X_A:

$$\mathbb{E}(X_A) = v_1(\mathbb{E}(t_A^+) - \mathbb{E}(t_A^-)) + \frac{1}{2} f \mathbb{E}(t_A^{-2})$$

$$= \frac{1}{2} f_0 \theta^2 \left(\frac{1 - e^{-r}}{r}\right) A^{-1} + O(A^{-3/2}) \tag{4.25}$$

$$\mathbb{E}(X_A^2) = \mathbb{E}(X_A^{+2}) + \mathbb{E}(X_A^{-2}) + 2\mathbb{E}(X_A^+)\, \mathbb{E}(X_A^-)$$

$$= 2\theta^2 v_1^2 A^{-1} + O(A^{-3/2}) \tag{4.26}$$

$$\mathbb{E}(X_A^{2k}) \leqslant 2^{2k-1} \mathbb{E}((X_A^+)^{2k}) + \mathbb{E}((X_A^-)^{2k})) \leqslant D_K A^{-k} \tag{4.27}$$

By using (4.17) and (4.25)–(4.27) we obtain that

$$\begin{aligned} &\text{(a)} \quad \leqslant \varepsilon^{-1} (D_4 D_2)^{1/2} K_A(t)\, A^{-3/2} \\ &\text{(b)} \quad = 2\theta^2 v_1^2 K_A(t)\, A^{-1} + K_A(t)\, O(A^{-3/2}) \end{aligned} \tag{4.13$'$}$$

Therefore, to conclude the proof of Lemma 4.1, we need only show that for all $T<\infty$ and $\varepsilon>0$

$$\lim_{A\to\infty} P(\{\sup_{t\in[0,T]} |K_A(t)\,A^{-1}-\tfrac{1}{2}t\theta^{-1}|>\varepsilon\})=0 \tag{4.28}$$

Let $T_n(t)=\sum_{i=0}^{n} t_{A,i}$ and $c=\frac{1}{2}\theta^{-1}$; be definition of $K_A(T)$ we obtain that

$$\begin{aligned}\{K_A(t)-cAT>A\varepsilon\}&\subseteq\{T_{K_A(T)}>\varepsilon A+cAT\}\\&\subseteq\{T_{K_A(T)}>T_{[\varepsilon A+cAt]}\}\subseteq\{t+t_{A,K_A(T)}>T_{[\varepsilon A+cAt]}\}\end{aligned} \tag{4.29}$$

where $[x]$ denotes the integer part of x. Note that by (4.24)

$$E(T_{[\varepsilon A+cAt]})=(\varepsilon A+cAt)(cA)^{-1}+O(A^{-3/2})=\varepsilon c^{-1}+t+O(A^{-1/2})$$

Then

$$\begin{aligned}&\{t+t_{A,K_A(t)}>T_{[\varepsilon A+cAt]}\}\\&\quad=\{t_{A,K_A(t)}>T_{[\varepsilon A+cAt]}-(t+\varepsilon c^{-1})+\varepsilon c^{-1}\}\\&\quad\subseteq\{|T_{[\varepsilon A+cAt]}-(t+\varepsilon c^{-1})|\geqslant\tfrac{1}{2}\varepsilon c^{-1}\}\\&\qquad\cup(\{|T_{[\varepsilon A+cAt]}-(t+\varepsilon c^{-1})|\geqslant 2\varepsilon c^{-1}\}\\&\qquad\cap\{t_{A,K_A(t)}<T_{[\varepsilon A+cAt]}-(t+\varepsilon c^{-1})+\varepsilon c^{-1}\})\\&\quad\subseteq\{T_{[\varepsilon A+cAt]}-(t+\varepsilon c^{-1})>\tfrac{1}{2}\varepsilon^{-1}\}\cup\{t_{A,K_A(t)}>\tfrac{1}{2}\varepsilon c^{-1}\}\end{aligned} \tag{4.30}$$

Moreover, since [cf. (4.11)]

$$\sum_{n=0}^{[cAt]+1} t_{A,n}-t\geqslant\sum_{n=0}^{[cAt]+1} t_{A,n}-\sum_{n=0}^{K_A(t)-1} t_{A,n}$$

we obtain in a similar way as in (4.30) that

$$\begin{aligned}&\{K_A(t)<cAt-A\varepsilon\}\\&\quad\subseteq\{T_{[cAt]+1}-t\geqslant T_{K_A(t)+[A\varepsilon]}-T_{K_A(t)-1}\}\\&\quad\subseteq\left\{\left|T_{[cAt]}-t>\frac{1}{3}\varepsilon c^{-1}\right\}\cup\left\{\sum_{n=K_A(t)}^{K_A(t)+[A\varepsilon]}(t_{A,n}-\varepsilon c^{-1})\right|>\frac{1}{3}\varepsilon c^{-1}\right\}\\&\qquad\cup\left\{\frac{1}{3}\varepsilon c^{-1}+t_{A,[cA]+1}>\frac{2}{3}\varepsilon c^{-1}\right\}\end{aligned} \tag{4.31}$$

By (4.30), (4.31), Chebychev's inequality with the fourth moment, and (4.24) we have

$$\begin{aligned} &P(\{|K_A(t)\,A^{-1}-ct|>\varepsilon\}) \\ &\quad\leqslant P(\{K_A(t)-cAt>A\varepsilon\})+P(\{K_A(t)<cAt-\varepsilon A\}) \\ &\quad\leqslant c_1 t^3 A^{-3}+c_2 A^{-3} \end{aligned} \tag{4.32}$$

Now, let $[0, T]=\bigcup_{n=1}^{N}\Gamma_n$, $\Gamma_n=[u_{n-1}, u_n]$, $u_0=0$, $u_n=T$, $|u_n-u_{n-1}|\leqslant\delta$. Then

$$\begin{aligned} &P(\{\sup_{t\in[0,T]}|K_A(t)\,A^{-1}-ct|>\varepsilon\} \\ &\quad\leqslant\sum_{n=1}^{N}P(\{\sup_{u\in\Gamma_n}|K_A(u)\,A^{-1}-cu|>\varepsilon\}) \\ &\quad\leqslant\sum_{n=1}^{N}[P(\{\sup_{u\varepsilon\Gamma_n}|K_A(u)\,A^{-1}-cu|>\varepsilon\}\cap\{(K_A(u_n)-K_A(u_{n-1}))\leqslant k_0\}) \\ &\quad\quad+P(\{|K_A(u_n)-K_A(u_{n-1})|>k_0\})] \end{aligned} \tag{4.33}$$

We first handle the second term on the rhs of (4.34). Since

$$K_A(u_n)-K_A(u_{n-1})\leqslant\inf\left\{k\geqslant K_A(u_{n-1}):\ \sum_{j=K_A(u_{n-1})}^{k}t_{A,j}\geqslant u_n-u_{n-1}\right\} \tag{4.34}$$

by homogeneity and noting that $|u_n-u_{n-1}|\leqslant\delta$, we have, therefore, that

$$\begin{aligned} P(\{K_A(u_n)-K_A(u_{n-1})>k_0\})&\leqslant P(\{K(\delta)>k_0\}) \\ &\leqslant P\left(\left\{\sum_{n=0}^{k_0-1}t_{A,n}<\delta\right\}\right) \\ &\leqslant e^{\alpha\delta}(\mathbb{E}(e^{-\alpha t_A}))^{k_0} \\ &\leqslant C_4 e^{\alpha\delta}(1+2\alpha/cA)^{-2k_0} \end{aligned} \tag{4.35}$$

To get the last inequality, we have used (4.16) and (4.22) with $e^{-\alpha t}$ instead of t.

Finally, since, for all $u\in\Gamma_n$, $|u-u_n|\leqslant\delta$, we obtain for the first term on the right of (4.33)

$$\begin{aligned} &\{\sup_{u\in\Gamma_n}|K_A(u)\,A^{-1}-cu|>\varepsilon\}\cap\{K_A(u_n)-K_A(u_{n-1})>k_0\} \\ &\quad\subseteq\{\sup_{u\in\Gamma_n}|(K_A(u_n)-K_A(u))\,A^{-1}|+|K_A(u_n)\,A^{-1}-cu_n|>\varepsilon-\delta c\} \\ &\quad\quad\cap\{K_A(u_n)-K_A(u_{n-1})<k_0\} \\ &\quad\subseteq\{|K_A(u_n)-cu_n|>\varepsilon-\delta c-k_0A^{-1}\} \end{aligned} \tag{4.36}$$

and this fields

$$P(\{\sup_{u\in\Gamma_n}|K_A(u)A^{-1}-cu|>\varepsilon\})$$
$$\leqslant P(\{|K_A(u_n)-cu_n|>\varepsilon-\delta c-k_0A^{-1}\})\leqslant C_5T^3A^3 \qquad (4.37)$$

The last inequality follows from (4.32), for δ small enough and A large enough. Then, choosing $\delta = A^{-3/2}$, $k_0 \geqslant 3$, and $\delta = A^{3/2}$, (4.28) easily follows from (4.34), (4.35), and (4.37). ∎

5. THE CONVERGENCE OF THE MECHANICAL PROCESS

In this section, we complete the proof of Theorem 2.1 by showing the closeness of the processes Q_A and $\hat{Q}_A$. This is the subject of the following proposition.

Proposition 5.1. Let $\hat{Q}_A$ be the process defined in Section 4; then, for all $T < +\infty$,

$$\lim_{A\to\infty} P(\{\sup_{t\in[0,T]}|Q_A(t)-\hat{Q}_A(t)|>\varepsilon\})=0$$

Proof. By Lemma 3.4, we need only show that

$$\lim_{A\to\infty} P(\Omega_A(T)\cap\{\sup_{t\in[0,T]}|Q_A(t)-\hat{Q}_A(t)|>\varepsilon\})=0$$

We may write [cf. (4.8)]

$$\begin{aligned}
&Q_A(t)-\hat{Q}_A(t)\\
&\quad=Q'_{A,0}+\left[Q_A(t)-Q_A\left(\sum_{j=0}^{N'_A(t)}\tau'_{A,j}\right)\right]\\
&\quad\sum_{\substack{j=1\\ j\in N_1}}^{N'_A(t)}(Q'_{A,j}-\hat{Q}_{A,j})+\sum_{\substack{j=1\\ j\in N_2}}^{N'_A(t)}(Q'_{A,j}-\hat{Q}_{A,j})-\sum_{n=0}^{K_A^<(t)}X_{A,n}+\sum_{j=K_A^>(t)}^{N'_A(t)}Q_{A,j}
\end{aligned} \qquad (5.1)$$

and

$$\begin{aligned}
&\{\sup_{t\in[0,T]}|Q_A(t)-\hat{Q}_A(t)|>\varepsilon\}\\
&\quad\subseteq\left\{\sup_{t\in[0,T]}\left|\sum_{\substack{j=1\\ j\in N_1}}^{N'_A(t)}(Q'_{A,j}-\hat{Q}_{A,j})\right|>\frac{\varepsilon}{5}\right\}\\
&\quad\cup\left\{\sup_{t\in[0,T]}\left|\sum_{\substack{j=1\\ j\in N_2}}^{N'_A(t)}(Q'_{A,j}-\hat{Q}_{A,j})\right|>\frac{\varepsilon}{5}\right\}\\
&\quad\cup\left\{Q'_{A,0}+\sup_{t\in[0,T]}\left|Q_A(t)-Q_A\left(\sum_{j=0}^{N'_A(t)}\tau_{A,j}\right)\right|>\frac{\varepsilon}{5}\right\}\\
&\quad\cup\left\{\sup_{t\in[0,T]}\left|\sum_{n=0}^{K_A^<(t)}X_{A,n}\right|>\frac{\varepsilon}{5}\right\}\cup\left\{\sup_{t\in[0,T]}\left|\sum_{j=K_A^>(t)}^{N'_A(t)}Q'_{A,j}\right|>\frac{\varepsilon}{5}\right\}
\end{aligned} \qquad (5.2)$$

We start by computing the probabilities of second and third sets on the right of (5.2). Since $\sup_{t\in[0,T]} |Q_A(t)|$ and $\sup_{t\in[0,T]} |\hat{Q}_A(t)|$ are bounded by $D_1 A T^2$, D_1 a finite constant, and since for all $j>0$,

$$|Q'_{A,j}| < 2v_1(\tau'^-_{A,j}+\tau'^+_{A,j}) + \tfrac{1}{2}f[\tau'^2_{A,j}+(\tau'^+_{A,j})^2]$$

$$|\hat{Q}_{A,j}| < 2v_1(\hat{\tau}^-_{A,j}+\hat{\tau}^+_{A,j}) + \tfrac{1}{2}f(\hat{\tau}^-_{A,j})^2$$

for $\alpha < \gamma \leqslant 1/20$ we have that

$$\begin{aligned}
&\mathbb{E}(|Q'_{A,j} - \hat{Q}_{A,j}|\; X_{\{j\in N_2\}\cap\Omega_A(t)}\\
&\quad\leqslant \mathbb{E}([|Q'_{A,j}| + |\hat{Q}_{A,j}|]\; X_{\{j\in N_2\}\cap\Omega_A(T)})\\
&\quad\leqslant D_1 A T^2 [P(\{\tau'^-_{A,j} > A^{-1+\gamma}\}\cap\Omega_A(T)) + P(\{\tau'^+_{A,j} > A^{-1+\gamma}\}\cap\Omega_A(T))\\
&\qquad + P(\{\hat{\tau}^-_{A,j} > A^{-1+\gamma}\}) + P(\{\hat{\tau}^+_{A,j} > A^{-1+\gamma}\})]\\
&\qquad + 2(2v_1 A^{-1+\gamma} + f A^{-2(1+\gamma)})\; P(\{j\in N_2\}\cap\Omega_A(T))\\
&\quad\leqslant D_4 A^{-3/2+3\gamma}
\end{aligned} \tag{5.3}$$

we get the last inequality by Lemma 3.4, Lemma 3.3, (3.50), and by observing that for all j and γ, $k>0$ [see (4.18), (4.24)]

$$P(\{\hat{\tau}^+_{A,j} > A^{-1+\gamma}\}) + P(\{\hat{\tau}^-_{A,j} > A^{-1+\gamma}\}) \leqslant 2D_3 k!\, A^{-\gamma k} \tag{5.4}$$

Therefore, using Lemma 3.3, Chebychev's inequality, and (5.3), we obtain

$$\begin{aligned}
&P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{\substack{j=1\\ j\in N_2}}^{N'_A(t)} (Q'_{A,j} - \hat{Q}_{A,j})\right| > \frac{\varepsilon}{5}\right\}\cap\Omega_A(T)\right)\\
&\quad\leqslant P(\{N_A(T) > rAT\})\\
&\qquad + 5\varepsilon^{-1}\sum_{j=1}^{[rAT]} \mathbb{E}(|Q'_{A,j} - \hat{Q}_{A,j}|\; X_{\{j\in N_2\}\cap\Omega_A(T)\cap\{j\leqslant N'_A(T)\}})\\
&\quad\leqslant D_5 A^{-1/2+3\gamma}
\end{aligned} \tag{5.5}$$

We shall now handle the third set on the rhs of (5.2). Since for all $t\in[0,T]$, $Q'_{A,0}$ and

$$\mathbb{E}\left(|Q'_{A,0}| + \left|\sup_{t\in[0,T]} Q_A(t) - Q_A\left(\sum_{j=0}^{N'_A(t)} \tau'_{A,j}\right)\right|\right)$$

as in (5.3), so that again by Chebychev's inequality

$$\begin{aligned}
&P\left(\left\{A'_{A,0}| + \left|\sup_{t\in[0,T]} Q_A(t) - Q_A\left(\sum_{j=0}^{N'_A(t)} \tau'_{A,j}\right)\right| > \varepsilon^{-1}/5\right\}\cap\Omega_A(T)\right)\\
&\quad\leqslant 10 D_4 \varepsilon^{-1} A^{-3/2+3\gamma}
\end{aligned}$$

It is more delicate to estimate the probabilities of the first and of the last two sets on the rhs of (5.2).

It is useful to define besides $\bar{E}$ a stronger property E.

Property *E*. We say that j as property E if the following two conditions hold:

E_a: during the interval $\Delta'_{A,j}$, the mechanical stick has not more than two collision with a fresh atom with $p^1 = +v_1$ and not more than two collisions with fresh atoms with $p^1 = -v_1$

E_b: the fresh atoms which collide with the mechanical stick during $\Delta'_{A,j}$ are the same as those which collide with the virtual stick and which define $\hat{Q}'^{+}_{A,j}$ and $\hat{Q}'^{-}_{A,j}$

Note that property E implies property $\bar{E}$.

We shall use the fact that the probability of E to fail is small. This is established in the next lemma, whose proof we postpone until the end of this section.

Lemma 5.2. Let E_j be the set of trajectories of which j has property E; then for all $j = 0,..., N'_A(T)$,

$$P(\{j \in N_1\} \cap (\Omega_A(T) \cap E_j^c) \leqslant C_8 A^{-1+4\gamma} \quad \blacksquare$$

By Chebychev's inequality it follows that

$$\begin{aligned}
&P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)} (Q'_{A,j} - \hat{Q}_{A,j})\right| > \frac{\varepsilon}{5}\right\} \cap \Omega_A(T)\right) \\
&\leqslant P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)} (Q'_{A,j} - \hat{Q}_{A,j})\,\chi_{E_j^c}\right| > \frac{\varepsilon}{15}\right\} \cap \Omega_A(T)\right) \\
&\quad + P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)} \left(Q'_{A,j} - \hat{Q}_{A,j} + v_1 \hat{t}_{j-1}\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. + \frac{f}{2}\hat{t}_{j-1}^2 - v_1\hat{t}_j + \frac{f}{2}\hat{t}_j^2\right)\chi_{E_j}\right| > \frac{\varepsilon}{15}\right\} \cap \Omega_A(T)\right) \\
&\quad + P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)} \left(-v_1\hat{t}_{j-1} + \frac{f}{2}\hat{t}_{j-1}^2 + v_1\hat{t}_j - \frac{f}{2}\hat{t}_j^2\right)\chi_{E_j}\right| > \frac{\varepsilon}{15}\right\} \cap \Omega_A(T)\right)
\end{aligned} \tag{5.6}$$

where $\hat{t}_j$ is the time defined in (3.44).

We shall start with the first term on the rhs of (5.6). As in (5.5), by replacing $\{j \in N_2\}$ with E_j^c, we have

$$P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)}(Q'_{A,j}-\hat{Q}_{A,j})\,\chi_{E_j^c}\right|>\frac{\varepsilon}{15}\right\}\cap\Omega_A(T)\right)$$
$$\leqslant P(\{N'_A(T)>rAT\}\cap\Omega_A(T))$$
$$+15\varepsilon^{-1}\sum_{j=1}^{[rAT]}\mathbb{E}(|Q'_{A,j}-\hat{Q}_{A,j}|\,\chi_{E_j^c}\cap\{j\leqslant N'_A(T)\}$$
$$\leqslant D_5A^{-1/2+3\gamma} \tag{5.7}$$

From the construction of $\hat{Q}'_{A,j}$, we obtain that on E_j

$$\hat{Q}'_{A,j}=Q'^{-}_{A,j}+v_1\hat{t}_{j-1}-\tfrac{1}{2}f\hat{t}^2_{j-1}+Q'^{+}_{A,j}-v_1u_j$$

with

$$u_j=(Q'^{+}_{A,j}-v_1\tau'^{+}_{A,j})/2v_1$$

Since $\hat{t}_j-u_j=f\hat{t}_j/4v_1$, it follows that

$$|Q'_{A,j}-\hat{Q}_{A,j}+v_1\hat{t}_{j-1}-\tfrac{1}{2}f\hat{t}^2_{j-1}-v_1\hat{t}_j+\tfrac{1}{2}f\hat{t}_j|\,\chi_{E_j}\leqslant f\hat{t}^2_j \tag{5.8}$$

and therefore by Chebychev's inequality

$$P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)}\left(Q'_{A,j}-\hat{Q}_{A,j}+v_1\hat{t}_{j-1}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\frac{1}{2}f\hat{t}^2_{j-1}-v_1\hat{t}_j+\frac{1}{2}f\hat{t}^2_j\right)\chi_{E_j}\right|>\frac{\varepsilon}{15}\right\}\cap\Omega_A(T)\right)$$
$$\leqslant P(\{N'_A(T)>rAT\}\cap\Omega_A(T))$$
$$+15\varepsilon^{-1}\sum_{j=1}^{[rAT]}E\left(\left|Q'_{A,j}-\hat{Q}+v^1\hat{t}_{j-1}\right.\right.$$
$$\left.\left.-\frac{1}{2}f\hat{t}^2_{j-1}-v^1\hat{t}_j+\frac{f}{2}\hat{t}^2_j\right|\chi_{E_j}\cap\{j\leqslant N'_A(T)\}\right)$$
$$\leqslant D_6A^{-1/2+3\gamma} \tag{5.9}$$

The last inequality follows from (3.48), (3.50), and (5.8).
Finally, for the last set on the rhs of (5.6)

$$\sum_{j\in N_1}^{N'_A(t)}\left(-v_1\hat{t}_{j-1}+\frac{f}{2}\hat{t}^2_{j-1}+v_1\hat{t}_j-\frac{1}{2}f\hat{t}^2_j\right)\chi_{E_j}$$
$$=\sum_{j\in N_1}^{N'_A(t)}\left(-v_1\hat{t}_{j-1}+\frac{1}{2}f\hat{t}^2_{j-1}\right)\chi_{E^c_{j-1}\cup\{(j-1)\in N_2\}}$$
$$+\left(v_1\hat{t}_j-\frac{1}{2}f\hat{t}^2_j\right)\chi_{E^c_{j-1}\cup\{(j+1)\in N_2\}}$$

and then, in the same way as in (5.7), we obtain that

$$P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{j\in N_1}^{N'_A(t)}\left(-v^1\hat{t}_{j-1}+\frac{f}{2}\hat{t}_{j-1}^2+v^1\hat{t}_j-\frac{f}{2}\hat{t}_j^2\right)\chi_{E_j}\right|>\frac{\varepsilon}{15}\right\}\cap\Omega_A(T)\right)$$
$$\leqslant D_5A^{-1+3\gamma} \tag{5.10}$$

We shall complete the proof of Proposition 5.1 by estimating the probabilities of the last two sets on the rhs of (5.2). Since both sets can be handled in the same way, we shall only deal with

$$\left\{\sup_{t\in[0,T]}\left|\sum_{n=0}^{K_A^<(t)}X_{A,n}\right|>\frac{\varepsilon}{5}\right\}$$

Let $\delta>0$; then from (4.27) and (4.28) we get

$$P\left(\left\{\sup_{t\in[0,T]}\left|\sum_{n=0}^{K_A^<(t)}X_{A,n}\right|>\frac{\varepsilon}{5}\right\}\right)$$
$$\leqslant P(\{\sup_{t\in[0,T]}K_A^<(t)>A^{1-\delta}\})$$
$$+P\left(\left\{\sum_{n=0}^{A^{1-\delta}}|X_{A,n}-\mathbb{E}(X_{A,n})|>\left(\frac{\varepsilon}{5}-A^{1-\delta}\mathbb{E}(X_{A,n})\right)\right\}\right)$$
$$\leqslant P(\{\sup_{t\in[0,T]}K_A^<(t)>A^{1-\delta}\})+D_8(\varepsilon-D_9A^{-\delta})^{-2}A^{-\delta} \tag{5.11}$$

where we have used Chebychev's inequality.

Hence we only need to show that the first term on the rhs of (5.11) goes to zero as $A\to\infty$. By definition of $K_A^<(t)$ [cf. (4.7a)]

$$\{\sup_{t\in[0,T]}K_A^<(t)>A^{1-\delta}\}$$
$$\subseteq\left\{\sup_{t\in[0,T]}\left|\sum_{j=1}^{N'_A(t)}(\hat{\tau}_{A,j}-\tau'_{A,j})+\left(t-\sum_{j=1}^{N'_A(t)}\tau'_{A,j}\right)\right|\geqslant\sum_{n=0}^{A^{1-\delta}}t_{A,n}\right\}$$
$$\subseteq\left\{\sum_{n=0}^{A^{1-\delta}}t_{A,n}\leqslant A^{-\delta}\right\}$$
$$\cup\left\{\sup_{t\in[0,T]}\left|\sum_{j=1}^{N'_A(t)}(\hat{\tau}_{A,j}-\tau'_{A,j})+\left(t-\sum_{j=1}^{N'_A(t)}\tau'_{A,j}\right)\right|\geqslant\lambda A^{-\delta}\right\} \tag{5.12}$$

Choosing $\delta<1/2$ and $\lambda<c\equiv 2\theta$, we obtain by (4.16) and (4.21) that the probability of the first set on the rhs of (5.12) is bounded by

$$P\left(\left\{\sum_{n=0}^{A^{1-\delta}} t_{A,n} \leqslant \lambda A^{-\delta}\right\}\right) \leqslant \exp(\xi\lambda A^{-\delta})]\ E(\exp[-\xi t_{A,0}])^{A^{1-\delta}}$$
$$\leqslant D_{16} \exp(\lambda A^{\delta})[\theta^{-1}/(\theta^{-1}+\xi A)]^{2A^{1-\delta}}$$
$$\leqslant D_{10} \exp[-(c-\lambda)\, A^{\delta}] \tag{5.13}$$

where $\xi = A^{2\delta}$.

Finally, note that for the second event on the rhs of (5.12)

$$\left\{\sup_{t\in[0,T]} \left|\sum_{j=1}^{N'_A(t)} (\hat{\tau}_{A,j} - \tau'_{A,j}) + \left(t - \sum_{j=1}^{N'_A(t)} \tau'_{A,j}\right)\right| > \lambda A^{-\delta}\right\}$$
$$\subseteq \left\{\sup_{t\in[0,T]} \left|\sum_{\substack{j=1\\ j\in N_1}}^{N'_A(t)} (\hat{\tau}_{A,j} - \tau'_{A,j})\right| > \frac{\lambda A^{-\delta}}{3}\right\}$$
$$\cup \left\{\sup_{t\in[0,T]} \left|\sum_{\substack{j=1\\ j\in N_2}}^{N'_A(t)} (\hat{\tau}_{A,j} - \tau'_{A,j})\right| > \frac{\lambda A^{-\delta}}{3}\right\}$$
$$\cup \left\{\sup_{t\in[0,T]} \left(t - \sum_{j=1}^{N'_A(t)} \tau'_{A,j}\right) > \frac{\lambda A^{-\delta}}{3}\right\} \tag{5.14}$$

and therefore we may bound the probabilities of these sets in the same way as for the corresponding sets for the increments $\hat{Q}_{A,j}$ and $Q'_{A,j}$. Proposition 5.1 follows easily from (5.2), (5.5), (5.7), (5.9), (5.10), and (5.14) for γ and δ sufficiently small. ▮

Proof of Lemma 5.2. In the following, $j \in N_1$ and $\omega \in \Omega_A(T)$. Since

$$E_j^c = \{E_a \text{ does not hold for } j\}$$
$$\cup \{E_a \text{ holds and } E_b \text{ does not hold for } j\} \tag{5.15}$$

for $j \in N_1$ there exists n such that

$$\Delta'^{+}_{A,j} = \Delta^{+}_{A,n} \qquad \text{or} \qquad \Delta'^{+}_{A,j} = \Delta^{+}_{A,n} \cup \Delta_{A,n+1}$$

Thus,

$$\{j \in N_1\} \cap \{E_a \text{ does not hold for } j\}$$
$$= \{\text{for some } n,\ \Delta'^{+}_{A,j} = \Delta^{+}_{A,n} \text{ and } E_a \text{ does not hold}\}$$
$$\cup \{\text{for some } n,\ \Delta'^{+}_{A,j} = \Delta^{+}_{A,n} \cup \Delta_{A,n+1} \text{ and } E_a \text{ does not hold}\} \tag{5.16}$$

Furthermore, recall that, for $j \in N_1$, during $\Delta'^{-}_{A,j}$ the mechanical stick can only collide with at most one fresh atom with $p^1 = -v_1$ and with at most two fresh atoms with $p^1 = v_1$. Therefore,

$$\begin{aligned}
&\Omega_A(T) \cap \{j \in N_1\} \cap \{\text{for some } n, \Delta'^{+}_{A,j} = \Delta^{+}_{A,n} \text{ and } E_a \text{ does not hold}\} \\
&\quad \subseteq (\{k_{2n-1} - k_{2(n-1)} = 0, \text{ the mechanical stick has at least two} \\
&\qquad \text{collisions with fresh atoms with } p^1 = v_1 \text{ during } \Delta^{+}_{A}, n \times \Delta'^{+}_{A,j}\} \\
&\qquad \cup \{k_{2n-1} - k_{2(n-1)} = 1, \text{ the mechanical stick collides with at} \\
&\qquad\quad \text{least one fresh atom with } p^1 = v_1 \text{ during } \Delta^{+}_{A,n} = \Delta'^{+}_{A,j}\}) \\
&\qquad \cap \{j \in N_1\} \cap \Omega_A(T) \\
&\quad \subseteq \{\text{the stick collides with a fresh atom with } p^1 = v_1 \text{ at some time} \\
&\qquad s \in [S'_{k2(j-1)}, S'_{k2(j-1)} + A^{-1+\gamma}] \text{ and with at least two fresh atoms with} \\
&\qquad p^1 = v^1 \text{ during } [s, s + A^{-1+\gamma}] \text{ where the stick has positive velocity}\} \\
&\qquad \cup \{n \text{ is not connected with any } k < n, \text{ the mechanical stick collides} \\
&\qquad\quad \text{with a fresh atom with } p^1 = -v_1 \text{ at some time} \\
&\qquad\quad s \in (S'_{k2(j-1)}, S'_{k2(j-1)} + A^{-1+\gamma}] \text{ and with at least one fresh atom} \\
&\qquad\quad \text{with } p^1 = v_1 \text{ during } [s, s + A^{-1+\gamma}] \text{ where the stick has positive} \\
&\qquad\quad \text{velocity}\} \\
&\qquad \cup (k(\mathrm{c})\, j \text{ for some } k < j, j \in N_1\} \\
&\qquad \cap \{\text{the stick has at least one collision with a fresh atom with } p^1 = v_1 \\
&\qquad\quad \text{during } [S'_{k_{2j-1}+1}, S'_{k_{2j-1}+1} + 2A^{-1+\gamma}] \text{ where it has positive} \\
&\qquad\quad \text{velocity}\}) \\
&\qquad \cup (\Omega_A(T) \cap (\{\tau'^{-}_{A,j} > A^{-1+\gamma}\} \cup \{\tau'^{+}_{A,j} > A^{-1+\gamma}\}))
\end{aligned}$$

For the third event observe also that

$$\begin{aligned}
&P(P(\{k(\mathrm{c})\, j \text{ for some } k < j, j \in N_1\} \\
&\quad \cap \{\text{the stick has at least one collision with a} \\
&\qquad \text{fresh atom } p^1 = v_1 \text{ during } [S'_{k_{2j-1}+1}, S'_{k_{2j-1}+1} + 2A^{-1+\gamma}] \\
&\qquad \text{where it has positive velocity}\} \mid Q_A(s), s < S'_{k_{2j-1}+1})) \\
&\quad = P(\{k(\mathrm{c})\, j \text{ for some } k < j, j \in N_1\}) \\
&\qquad \times P(\{\text{the stick has at least one collision with a} \\
&\qquad\qquad \text{fresh atom with } p^1 = v_1 \text{ during } [S'_{k_{2j-1}+1}, S'_{k_{2j-1}+1} + 2A^{-1+\gamma}] \\
&\qquad\qquad \text{where it has positive velocity}\} \mid Q_A(s), s < S'_{k_{2j-1}+1}))
\end{aligned}$$

and note that $S'_{k_{2j-1}+1}$ is a Markov time.

Thus, from Lemma 3.1 with $k = 1$, (3.40) with $n_0 = 2$, (5.3), and by estimating the first three events above with "Poisson domination," we obtain that

$$
\begin{aligned}
P(\{j\in N_1\} &\cap \Omega_A(T) \\
&\cap \{\text{for some } n,\ \Delta'^{+}_{A,j}=\Delta^{+}_{A,n} \text{ and } E_a \text{ does not hold}\}) \\
&\leqslant \frac{\rho l v_1}{2} A^{-1+\gamma}\left[\frac{\rho l f}{2}(2A^{-1+\gamma})^2\right]+\frac{\rho l f}{2}A^{-2+2\gamma}\left[\frac{\rho l f}{2}(2A^{-1+\gamma})^2\right] \\
&\quad + D_5 A^{-1/2-2\alpha}\left[\frac{\rho l f}{2}(2A^{-1+\gamma})^2\right]+(C_4 A^{-1+4\gamma}+D_2 A^{\gamma} e^{-\rho_0 l_0 v^1 A^{\gamma}}) \\
&\leqslant D'_5 A^{-1+4\gamma}
\end{aligned}
\tag{5.17}
$$

Moreover,

$$
\begin{aligned}
\{j\in N_1\} \cap \Omega_A(T) \cap \{&\text{for some } n,\ \Delta'^{+}_{A,j}=\Delta^{+}_{A,n}\cup\Delta_{A,n+1},\ E_a \text{ does not hold}\} \\
\subseteq \{&\text{for some } n,\ \Delta'^{+}_{A,j}=\Delta^{+}_{A,n}\cup\Delta_{A,n+1},\ k_{2n-1}-k_{2(n-1)}=0, \\
&\text{the mechanical stick has at least one collision with a fresh atom} \\
&\text{with } p^1=-v_1 \text{ during } \Delta^{-}_{A,n+1} \text{ or at least with fresh atom with} \\
&p^1=v_1 \text{ during } \Delta^{+}_{A,n}\cup\Delta_{A,n+1}\} \\
\cup \{&\text{for some } n,\ \Delta'^{+}_{A,j}=\Delta^{+}_{A,n}\cup\Delta_{A,n+1},\ k_{2n-1}-k_{2(n-1)}>0\}
\end{aligned}
$$

The probabilities of both these events may be handled in a similar way as before. That the last event has sufficiently small probability can also be read off from (3.48) of Lemma 3.4 and (3.59)–(3.61) of Lemma 3.5. Therefore, by (5.17), we obtain that

$$
P(\{j\in N_1\}\cap\Omega_A(T)\cap\{\text{(a) of property } E \text{ does not hold}\})\leqslant D_{11}A^{-1+4\gamma} \tag{5.18}
$$

For the second event on (5.16), suppose that E_b does not hold for the increments $\hat{Q}'^{-}_{A,j}$ and $Q'^{-}_{A,j}$.

Let $d(t)$ denote the distance between the mechanical and the virtual stick at time $S_{k_{2n}}+\hat{t}_n+t$. It is clear that to treat events for which E_b does not hold, we shall need bounds on $d(t)$. Let n be such that $\Delta'^{-}_{A,j}=\Delta^{-}_{A,n+1}$; by (3.60)–(3.62) of Lemma 3.5 it suffices to consider two cases: either $n(\mathrm{c})(n+1)$, $k_{2n+1}-k_{2n}=1$, $k_{2n-1}-k_{2(n-1)}=0$ and n, $(n-1)$, and $(n-2)$ are not connected, or $(n+1)$ is not connected with any $k<(n+1)$.

Note that the virtual stick and the mechanical stick start at time $S_{k_{2n}}+\hat{t}_n$ at the same point, with, however, different velocities ($-v_1$ for the virtual stick and $-v_1+f\hat{t}_n$ for the mechanical stick).

If $n(\mathrm{c})(n+1)$, then during $\Delta_{A,n+1}$ the mechanical stick may recollide, but its postcollision velocity in a recollision is bounded by $-v_1+f\tau_{A,n}$ [cf. (3.21)]. Therefore, we have

$$
d(t)\leqslant f(\tau_{A,n}\hat{t}_n)\, t \qquad \text{for all} \quad 0\leqslant t\leqslant \hat{\tau}^{-}_{A,n+1} \tag{5.19}
$$

If, furthermore, n is connected with $(n+1)$, then we have that

$$d(t) \leqslant f\hat{t}_n t \qquad \text{for all} \quad 0 \leqslant t \leqslant \hat{\tau}_{A,n+1} \wedge (\tau'_{A,n+1} - \hat{t}_n) \tag{5.20}$$

If an atom collides at time $S_{k_{2n}} + \hat{t}_n + t$ with one of the two sticks and if it collides with the other stick, say, at time $S_{k_{2n}} + \hat{t}_n + t + \sigma$, then

$$\sigma \leqslant \begin{cases} d(t)/ft & \text{if} \quad p^1 = -v_1 \\ d(t)/v_1 & \text{if} \quad p^1 = v_1 \end{cases} \tag{5.21}$$

Therefore, if E_b does not hold, then one of the following three cases may occur:

(i) A fresh atom collides with one of the two sticks in a region $I_y \subset [0, y] \cup [l-y, l]$ of length $y \leqslant v_2 \sigma$.

(ii) The first collision of the mechanical stick is at time $S_{k_{2n}} + \hat{t}_n + s$ for some s and is with a fresh atom with $p^1 = -v_1$. The first collision of the virtual stick is at time $S_{k_{2n}} + \hat{t}_n + t$ for some t and happens with a fresh atom with $p^1 = v_1$. If (i) does not occur and if the fresh atom with $p^1 = v_1$ collides also with the mechanical stick, say, at time $S_{k_{2n}} + \hat{t}_n + w$, then

$$|t - s| \leqslant \frac{d(s)}{fs}$$

Otherwise, the atom which collides first with the mechanical stick would also collide first with the virtual stick and

$$|s - w| \leqslant |s - t| + |w - t| \leqslant \frac{d(s)}{fs} + \frac{d(t)}{v_1} \tag{5.22}$$

(iii) The mechanical stick collides with a fresh atom with $p^1 = -v_1$ at time $S_{k_{2n}} + \hat{t}_n + s$ at the point $y \in [0, l]$ and the same atom "collides" with the virtual stick at the point $y' \in [0, l]$ and at time $S_{k_{2n}} + \hat{t}_n + t$, $t > s$. In this situation, if E_b does not hold, then either (i) occurs for the collision with a fresh atom with $p^1 = v_1$ or only one of the two sticks recollides [recall that a recollision for the virtual stick was defined to be nonmechanical; cf. (4.1)]. For this latter situation, note that if $p^2 < 0$ ($p^2 > 0$), then

$$t + y'/v_2 = s + y/v_2 \qquad (t + (l - y')/v_2 = s + (l - y)/v_2)$$

Let τ stand for either one of the times $t + y'/v_2$ or $s + y/v_2$ ($(l - y')/v_2$ or $s + (l - y)/v_2$). Let $\bar{d} = f(\tau'^{-2}_{A,j})^2$, where $\bar{d}$ is a bound on the sum of the distances between the trajectories of the two sticks and of the trajectories of the mechanical stick and of the atom which collided at time $S_{k_{2n}} + \hat{t}_n + s$ during

$\Delta^-_{A,n+1}$. Therefore, if a fresh atom with both $p^1 = v_1$ collides with both sticks, the virtual and the mechanical, say, at times $S_{k_{2n}} + \hat{t}_n + w_1$ and $S_{k_{2n}} + \hat{t}_n + w_2$, respectively, then $|w_1 - w_2| < \bar{d}/v_1$ and if only one of the sticks recollides, then $S_{k_{2n}} + \hat{t}_n + w_1$ or $S_{k_{2n}} + \hat{t}_n + w_2$ certainly belongs to the interval $[S_{k_{2n}} + \hat{t}_n - \bar{d}/v_1, S_{k_{2n}} + \hat{t}_n + \tau]$, for otherwise both sticks would recollide. Using this, (5.20), (5.21), and (3.59), we obtain that

$$\begin{aligned}
&P(\{\Delta'^-_{A,j} = \Delta^-_{A,(n+1)} \text{ for some } n, (n+1) \text{ is not connected with any}\\
&\quad k < n+1 \text{ and one of the three of (5.22) does not hold}\})\\
&\leqslant P(\{\tau'^+_{A,j-1} > A^{-1+\gamma}\} \cap \Omega_A(T))\\
&\quad + P(\{\tau'^-_{A,j} > A^{-1+\gamma}\} \cap \Omega_A(T)) + P(\{\tau^-_{A,j} > A^{-1+\gamma}\})\\
&\quad + 4\underbrace{\left[\frac{\rho f}{4} A^{-2+2\gamma}\left(v_2 \frac{f}{2v_1} A^{-2+2\gamma}\right) + \rho v_1 A^{-1+\gamma}\left(v_2 \frac{f}{v_1}\frac{f}{2v_1} A^{-3+3\gamma}\right)\right]}_{(5.22)(\mathrm{i})}\\
&\quad + \frac{1}{4}\rho l f \rho l v^1 A^{-2+2\gamma}\underbrace{\left[\frac{f}{2v_1} A^{-2+2\gamma} + \frac{f}{v_1}\frac{f}{2v_1} A^{-2+2\gamma}\left(A^{-1+\gamma} + \frac{f}{2v_1} A^{-2+2\gamma}\right)\right]}_{(5.22)(\mathrm{ii})}\\
&\quad + \underbrace{2\rho l f A^{-2+2\gamma} \rho l v_1 \frac{f}{2v_1} A^{-2+\gamma}}_{(5.22)(\mathrm{iii})} \leqslant D_{12} A^{-1+4\gamma}
\end{aligned} \tag{5.23}$$

The last inequality follows from (5.3), Remark 3.2 for $n_0 \geqslant 2$, and (5.5) for $k > 1/\gamma$.

Note that on

$$\begin{aligned}
\{&\text{for some } n, \Delta'^-_{A,j} = \Delta^-_{A,n}, n(\mathrm{c})(n+1),\\
&k_{2n+1} - k_{2n} = 1, k_{2n-1} - k_{2(n-1)} = 0,\\
&\text{is not connected with any } k < n\}
\end{aligned}$$

only (i) can fail. Therefore, by using (5.19) to bound σ in (5.21), we have

$$\begin{aligned}
&P(\{\text{for some } n, \Delta'^-_{A,j} = \Delta^-_{A,n}, n(\mathrm{c})(n+1),\\
&\quad k_{2n+1} - k_{2n} = 1, k_{2n-1} - k_{2(n-1)} = 0,\\
&\quad n \text{ is not connected with any } k < n\}\\
&\qquad \cap \{\text{one of the three conditions of (5.13) does not hold}\})\\
&\quad \leqslant P(\{\text{the mechanical stick collides with a fresh atom with } p^1 = v_1 \text{ at}\\
&\qquad \text{time } s \in \Delta^-_{A,n} \text{ and with at least one fresh atom with } p^1 = -v_1\\
&\qquad \text{during } [s, s + \tfrac{1}{2} f/v_1 v_2 A^{-1+\gamma}]\}
\end{aligned}$$

$$
\begin{aligned}
&\cap \{\text{condition (i) of (5.22) with} \\
&\quad u = f/v_1(A^{-1+\gamma} + f/2v_1 A^{-2+2\gamma})\, A^{-1+\gamma} \text{ does not hold for} \\
&\quad Q'^{-}_{A,j}\,(=Q^{-}_{A,n+1}) \text{ of } \hat{Q}^{-}_{A,j}\}) \\
&+ P(\Omega_A(T) \cup (\{\tau^{-}_{A,n} > A^{-1+\gamma}\} \\
&\cup \{\tau^{-}_{A,n+1} > A^{-1+\gamma}\} \cup \{\tau^{-}_{A,n} > A^{-1+\gamma}\})) \\
\leqslant\, & \rho l v_1 A^{-1+\gamma} \rho l v_1 (f/2v_1 v_2)\, A^{-1+\gamma} \\
\leqslant\, & 2\rho v_1 A^{-2+2\gamma}(f v_2/v_1)[A^{-1+\gamma} + (f/2v_1)\, A^{-2+2\gamma}] \\
&+ (2D_3 A^{-1} + 2C_4 A^{-1+4\gamma}) \\
\leqslant\, & D_{13} A^{-1+5\gamma}
\end{aligned}
\tag{5.24}
$$

To obtain the second estimate we have used (5.4), Lemma 3.3 with $n_0 = 2$, and the obvious "Poisson domination" similarly as in (5.23).

Finally, we obtain that

$$
\begin{aligned}
&P(\{E_a \text{ holds and } E_b \text{ does not hold for } Q'^{-}_{A,j} \text{ and } \hat{Q}_{A,j}\} \cap \Omega_A(T)) \\
&\quad\leqslant P(\{(n-1), n, (n+1) \text{ are connected}\} \cap \Omega_A(T)) \\
&\qquad + P(\{n(\mathrm{c})(n+1), n \text{ is not connected with any } k < n \text{ and either} \\
&\qquad\qquad k_{2n} - k_{2n} > 2 \text{ or } k_{2n-1} - k_{2(n-1)} \geqslant 1\} \\
&\qquad \cap \Omega_A(T)) \\
&\qquad + P(\{\text{for some } n,\ \Delta'^{-}_{A,j} = \Delta^{-}_{A,n+1},\ n(\mathrm{c})(n+1),\ k_{2n+1} - k_{2n} = 1, \\
&\qquad\qquad k_{2n-1} - k_{2(n-1)} = 0,\ n \text{ is not connected with any } k < n, \text{ one} \\
&\qquad\qquad \text{of the three conditions of (5.22) does not hold}\}) \\
&\qquad + P(\{\text{for some } n,\ \Delta'^{-}_{A,j} = \Delta^{-}_{A,n+1},\ (n+1) \text{ is not connected with any} \\
&\qquad\qquad k < n \text{ and one of the three conditions (5.22) does not hold}\} \\
&\qquad \cap \Omega_A(T)) \\
&\quad\leqslant C_8 A^{-1+\beta} + D_8 A^{-1} + (D_{12} + D_{13})\, A^{-1+5\gamma}
\end{aligned}
\tag{5.25}
$$

The last inequality follows from (3.60), (3.61), and (3.63) of Lemma 3.5 and (5.23), (5.24).

To complete proof of the lemma, we now consider the set

$$
B \equiv \{E_a \text{ holds and } E_b \text{ is violated only for } Q'^{+}_{A,j} \text{ and } \hat{Q}^{+}_{A,j}\} \cap \Omega_A(T) \tag{5.26}
$$

Let us denote

$$
\bar{d} = \sup_{0 \leqslant s \leqslant \tau^{-}_{A,j}} d(s) \qquad \text{and} \qquad \bar{\sigma} = \bar{d}/v_1
$$

Roughly speaking, $\bar{\sigma}$ is the maximum time that a fresh atom with $p^1 = v_1$, which collided with one stick, needs to reach the other.

First, we observe that

$$\begin{aligned}
B \subseteq (&\{\text{the virtual stick collides with at least one atom during} \\
&\quad S'_{k2(j-1)} + \hat{t}_{j-1} + \hat{\tau}^-_{A,j},\ S'_{k2(j-1)} + \hat{t}_{j-1} + \hat{\tau}^-_{A,j} + \bar{\sigma}]\} \\
&\cup \{\text{the mechanical stick collides with at least one fresh atom} \\
&\quad \text{during } (S'_{k2(j-1)} + \tau'^-_{A,j},\ S_{k2(j-1)} + \tau'^-_{A,j} + \bar{\sigma}]\}) \\
&\cup (B \cap \{\text{the virtual stick does not collide during} \\
&\quad S'_{k2(j-1)} + \hat{t}_{j-1} + \hat{\tau}_{A,j},\ S'_{k2(j-1)} + \hat{t}_{j-1} + \hat{\tau}^-_{A,j} + \bar{\sigma}]\} \\
&\cap \{\text{the mechanical stick does not collide with a fresh atom} \\
&\quad \text{during } (S'_{k2(j-1)} + \tau'^-_{A,j},\ S'_{k2(j-1)} + \tau'^-_{A,j} + \bar{\sigma}]\} \\
\equiv B_1 \cup B_2 &
\end{aligned} \tag{5.27}$$

For this, note that in the event B_1 one of the two sticks may change its velocity prior to the time where the negative increment of the other stick ends.

Since $\bar{\sigma} \leqslant 2f(\tau'^-_{A,j} \wedge \hat{\tau}^-_{A,j})^2$, we obtain, similarly as in (5.23) and (5.24), that the probability of B_1 is bounded by $D_{14} A^{-1+\gamma}$.

Now, to handle the event B_2, we need some more detailed information on $d(t)$. We distinguish the situations where, for some n, (1) $\Delta'^+_{A,j} = \Delta^+_{A,n}$ or (2) $\Delta'^+_{A,j} = \Delta^+_{A,n} = \Delta^+_{A,n} \cup \Delta_{A,n+1}$.

In the first case again we distinguish two situations where during $\Delta^+_{A,n}$ the mechanical stick *does* or *does not* collide with a fresh atom with $p^1 = v_1$. We start with the latter situation. Let $d_0(t) = d(\hat{\tau}^-_{A,j} + \bar{\sigma} + t)$; then

$$d_0(t) \leqslant d(\hat{\tau}^-_{A,j} + \bar{\sigma}) + f\bar{\sigma}t \tag{5.28}$$

because $|\tau^-_{A,j} - \hat{t}_{j-1} - \hat{\tau}^-_{A,j}| \leqslant \bar{\sigma}$ and therefore the velocities of the two sticks at time $S'_{k2(j-1)} + \hat{t}_{j-1} + \hat{\tau}'^-_{A,j} + \bar{\sigma}$ cannot differ more than $f\bar{\sigma}$.

In the second situation, the difference in the velocities of the two sticks may be bounded by $f(\bar{\sigma} + t)$ and hence

$$d_0(t) \leqslant d(\hat{\tau}^-_{A,j} + \bar{\sigma}) + f(\sigma + t)\, t \qquad \text{for all} \quad 0 \leqslant t \leqslant \hat{\tau}^+_{A,j} \tag{5.29}$$

Since $|\tau'^-_{A,j} - \hat{\tau}^-_{A,j} - t_{j-1}| \leqslant \bar{\sigma}$, we have a trivial bound on $d(\hat{\tau}^-_{A,j} + \bar{\sigma})$ which is given by the deviation of the parabola from the tangent, namely

$$d(\hat{\tau}^-_{A,j} + \bar{\sigma}) \leqslant \frac{1}{2} f(2\bar{\sigma})^2 = 2f\bar{\sigma}^2 \tag{5.30}$$

If all the atoms that collide with the mechanical stick during $\Delta'^{+}_{A,j}$ collide also with the virtual stick, then

$$d_0(t) \leqslant 2f\bar{\sigma}^2 + f\bar{\sigma}t, \qquad 0 \leqslant t \leqslant \hat{\tau}^{+}_{A,j} \tag{5.31}$$

On $\Omega_A(t) \cap \{\tau'_{A,j} \leqslant A^{-1+\gamma}\} \cap \{\hat{\tau}'_{A,j} \leqslant A^{-1+\gamma}\}$ we obtain by (5.19) and (5.20) bound on $d_0(t)$ which are used in he inclusion below to give essentially a detailed description of B_2 [see (5.26) and (5.27)]

$$B_2 \cap \Omega_A(T) \cap \{\tau'_{A,j} \leqslant A^{-1+\gamma}\} \cap \{\hat{\tau}'_{A,j} \leqslant A^{-1+\gamma}\}$$

$$\subseteq \Bigg(\Big\{\text{the mechanical stick collides with a fresh atom with } p^1 = v_1 \text{ at some time } s \in [S'_{k_{2j-1}+1}, S'_{k_{2j-1}+1} + A^{-1+\gamma}] \text{ when it has positive velocity}\Big\}$$

$$\cap \Bigg(\Big\{\text{the mechanical stick collides with a fresh atom at some time } u \in [s, s + A^{-1+\gamma}] \text{ in a region } I_y \subseteq [0, y] \cup [l - y, l] \text{ of length } y \leqslant \frac{fv_2}{v_1}\left(\frac{6f}{v_1} A^{-3+3\gamma} + A^{-2+2\gamma}\right)\Big\}$$

$$\cup \Big\{\text{the virtual stick collides with an atom in a region } I_y \subseteq [0, y] \cup [l - y, l] \text{ of length } y \leqslant \frac{fv_2}{v_1}\left(\frac{6f}{v_1} A^{-3+3\gamma} + A^{-2+2\gamma}\right)\Big\}\Bigg)\Bigg)$$

$$\cup \Big\{\text{the mechanical stick collides with a fresh atom in a region } I_y \subseteq [0, y] \cup [l - y, l] \text{ of length } y \leqslant \frac{2v_2 f^2}{v_1^2} A^{-2+2\gamma}\left(\frac{4f}{v_1} A^{-2+2\gamma} + A^{-1+\gamma}\right)\Big\}$$

$$\cup \Big\{\text{the virtual stick collides with an atom in a region } I_y \subseteq [0, y] \cup [l - y, l] \text{ of length } y \leqslant \frac{2v_2 f}{v_1} A^{-2+2\gamma}\left(\frac{4f}{v_1} A^{-2+2\gamma} + A^{-1+y}\right)\Big\} \tag{5.32}$$

Similarly as in (5.24) and (5.23), we obtain from this

$$P(\{E_a \text{ and } E_b \text{ hold for } Q'^{-}_{A,j} \text{ and } \hat{Q}^{-}_{A,j},\; E_b \text{ does not hold for } Q'^{+}_{A,j} \text{ and } \hat{Q}^{+}_{A,j}\} \cap \Omega_A(T)) \leqslant D_{14} A^{-1+\gamma} \tag{5.33}$$

Then (5.18), (5.25), and (5.33) yield the lemma.

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